DISCRETE ANALYTIC FUNCTIONS AND q-HYPERGEOMETRIC SERIES

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CERTIFICATE

Certified that the dissertation entitled **"Discrete analytic functions and q - hypergeometric series"** has been written under my guidance and supervision in the department of Applied Mathematics, Aligarh Muslim University, Aligarh. To the best of my knowledge and belief, the work included as such in this dissertation has not been submitted to any other University or Institution for the award of a degree.

This is to further certify that **Mr. Shakeel Ahmad Alvi** has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of Aligarh Muslim University, Aligarh.

MAKhan

Dated: 31.12.1992

(DR. MUMTAZ AHMAD KHAN)
SUPERVISOR

In the memory of my father

Dedicated to my mother,
(Mrs.) Rabia Khatoon

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PREFACE

Discrete analytic function theory is concerned with a study of functions defined only at certain lattice points in the complex plane. As it is suitable for treating ordinary difference functions, the lattice of definition is usually taken to be the set of gaussian integers.

Since its comparatively recent beginnings in 1941 the subject has been extensively developed by numerous writers. The resulting theory has much in common with the theory of analytic functions of a continuous complex variable and of course has many distinguishing features.

Scattered results on monodiffric functions, geometeric or q - difference functions, discrete q - hypergeometric functions and discrete functions on radial lattice necessitate a unified study of these functions. With this in view the present dissertation is an attempt to embody the results obtained in these areas from time to time by different authors.

The dissertation comprises five chapters. The first chapter covers a comprehensive account of the historical origin of the various theories of discrete analytic functions and the upto date developments made in these areas. This chapter also contains various definitions, notations and results used in this dissertation. The chapter also briefly introduces monodiffric functions, q - analytic functions, discrete q - hypergeometric functions and discrete functions on a radial lattice.

In II chapter the theory of monodiffric functions given by Isaacs have been discussed. Various properties of monodiffric functions, summation and monodiffric polynomials are given in this chapter.

In III chapter a thorough survey has been made of q - analytic functions due to Harman. Besides discussing elementary properties of q - analytic functions, the chapter also contains discrete line integral, discrete integration of q - analytic functions, analogue of Cauchy's Integral formula, discrete analytic continuation, continuation from the axis, multiplication of q - analytic functions, discrete powers, discrete exponential and trigonometric functions, discrete functions $(z-z_0)^{(a)}$ and series representation of q - analytic functions.

Chapter IV concerns the study of discrete q - hypergeometric functions introduced by Khan and gives its elementary properties and integral representations.

The V and final chapter is devoted to an exposition of discrete functions on a radial lattice and gives discrete analytic continuation and multiplication, discrete power functions and discrete power series representation of such functions.

I CHAPTER

INTRODUCTION

The present dissertation is a survey of discrete analytic functions defined on certain lattice points in the complex plane. In discrete function theory, the differential operator of the classical complex analysis is replaced by a suitable difference operator.

1. PRELIMINARIES. Functions of a complex variable can be classified in the following way. If

$$\lim_{r \to 0} \frac{f(z + re^{i\theta}) - f(z)}{r\,e^{i\theta}}$$

is dependent on θ, then the function is said to be 'polygenic'. If the limit is single-valued (independent of θ), the function is 'monogenic'. The study of functions of the latter type has of course developed into the wide field of analytic function theory of a continuous complex variable.

Discretisation of scientific models was initiated much earlier in applied mathematics than the study of discrete analyticity. In classical physics, phenomena of discrete character are usually treated as continua by a 'smoothing out' process. The powerful tools of analysis and differential equations are then applied Ruack [96] and Heisenberg [67,68] are pioneers of the principle of discretisation scientists felt dissatisfied by the over-emphasis of the continuum structure imposed on scientific models. The important difference between continuum and discrete structures is that infinitesimal is not considered in the latter. In discrete theory, the limit of a quotient of infinitesimals of the continuum structure is replaced by a quotient of finite quantities. Ruack [96] argued that 'the differential character of the principal equations of physics implies that physical systems are governed by laws which operate with a precision beyond the limits of verification by experiment. This appears undesirable from an axiomatic stand point'. He suggested that more emphasis should be given to the use of difference calculus in physics. In the classical finite difference theory, functions which are often defined on only a discrete set of points are usually treated as functions of a continuous variable. The established theories of analytic function theory can then be applied.

The important aspects are that the fundamental equations must be capable of describing every feature of the experiment and must not introduce extraneous or undesirable features.

Discrete hodon and chronon are introduced in Physics in recent times. This shows an interest from the side of scientists towards discretisation. Still, there is a task before the scientist to overcome. The differential equations are to be recasted in the form of difference equations.

In Margenau's [84] words, 'A word might be said about the reason why physicists are often reluctant to accept discreteness. If it were to be established as the ultimate property of time and space, one or the another of two drastic changes in the theoretical description of nature would have to take place. One is the recasting of all equations of motion in the form of difference equations instead of differential equations, and this is most unpalatable because of the mathematical difficulties attending the solution of difference equations. The other possible modification would involve the elimination of time and space coordinates from scientific description'.

Heisenberg is a powerful advocate of this. To simplify the problem, finite geometries of Veblen and others can be utilized or a continuous space time of the Minkowski form in which the events from a discrete lattice may be recommended.

The most general form of a lattice is a sequence of complex numbers, preferably a dense subset which is also countable. Accepting the postulate of rational description in Physics, the lattice of rational points in the complex plane: $\{(p,q);\ p,q \in Q$, the set of rational numbers$\}$ will be the best choice to build a discrete function theory.

In the earliest works of discrete function theory, the arithmetically spaced sequence, in particular the Gaussian integers was considered. Later in the beginning of two decades back, a function theory was developed on the set of geometrically spaced sequence. No work is done so far in the general set.

Now discrete function theory has grown to an established branch of Mathematics. The important problem is as E.T. Bell puts, 'A major task of Mathematics today is to harmonise the continuous and the discrete to include them in one comprehensive Mathematics and to eliminate obscurity from both'. Again a major taks of discrete analysts is the unification of known theories.

2. HISTORICAL DEVELOPMENTS. The theory of discrete functions has its beginning from 1941 through a distinguishing paper of R.P. Isaac [69] who modified the concept of monogeneity and introduced the notion of a 'monodiffric function', i.e. one that satisfies,

$$f(z+1) - f(z) = \frac{f(z+i) - f(z)}{i} \; ; \; i = \sqrt{-1} \qquad \ldots\ldots (1.2.1)$$

Using this concept of a difference quotient instead of a derivative, Isaacs constructed a theory for functions defined only on the set of Gaussian integers (points of the form $m + in$; m,n integers). Isaacs [69,70] in fact defined two types of discrete functions. Those satisfying (1.2.1) he termed 'monodiffric functions of the first kind'. Functions satisfying

$$f(z+1) - f(z-1) = \frac{f(z+i) - f(z-i)}{i} \qquad \ldots\ldots (1.2.2)$$

he called 'monodiffric functions of the second kind'. In each case the domain of definition of the function was the set of square lattice points - the Gaussian integers. He studied integration, residues, discrete powers and polynomials. Two of the major difficulties in discrete function theory are (1) the usual product of two discrete analytic functions in a domain is not discrete analytic in that domain and (2) the usual power of z are not discrete analytic in any domain in the discrete space. Isaacs himself realized these aspects and introduced the analogues.

In 1944 Ferrand [42] introduced the idea of a 'preholomorphic' function by means of the diagonal quotient equality

$$\frac{f(z+1+i) - f(z)}{1+i} = \frac{f(z+i) - f(z+1)}{i-1} \qquad \ldots\ldots (1.2.3)$$

which is essentially equivalent to the definition of functions of the second kind (1.2.2) given by Isaacs.

The development in discrete function theory, was slow for more than a decade from Ferrand's work, though Terracini and Romanov contributed in this decade to discrete function theory. The awakening was made by R.J. Duffin [32] in 1956. He [32-38] modified Ferrand's theory and extended the results to the realm of Applied Mathematics by discussing operational calculus and Hilbert transform. Pioneers of his school of discrete function theory are Duris [36,37], Rohrer [39], Peterson [38] and Kurowski [79-82]. Duffin [33] introduced rhombic lattice to develop potential theory. He also studied Yukawa potential theory in the discrete space of Gaussian integers [34]. Duffin and Duris [37] studied discrete product and discrete partial differential equations.

The Russian school of discrete function theory of which the leading names are Abdullaev [3-6], Babadanov [4-6], Chumakov [28], Silic [97] and Fuksman [44], has improved the theory by introducing different lattice, construction of a discrete analytic function and so on. In particular, Chumakov [28] developed semi-discrete function theory and Silic [97] investigated physical models in discrete function theory.

Hayabara [59-62], Deeter and Lord [30] developed operational calculus for discrete functions.

The school led by Deeter, whose distinguishing figures are Berzsenyi [16-18], Perry and Mastin [31] has studied discrete functions in Isaac's direction. Perry studied generalized discrete functions.

Abdullaev, Babadzanov and Hayabara developed discrete theory of higher dimensions. Kurowski [79-82] introduced a function theory in the semi-discrete lattice. Transform techniques were analysed by many like Duffin [32,35] and Badnar [13]

Mastin [31], Ferrand [42] and Isaacs [69,70] constructed theories of conformal representation. Tu [108-113] discussed discrete derivative equations and gave a generalisation of monodiffric functions. The discrete theory was extended by Hundhausen to harmonic analysis. Deeter [29] and Berzsenyi [18] gave comprehensive bibliography of discrete function theory.

All the works so far explained are mainly in the set of Gaussian integers. Harman [52-58] developed a discrete function theory in the geometric lattice, by utilizing the q-difference theory developed by Jackson, Hahn and Abdi. Differentiation, integration, convolution product, polynomial theory and conformal mapping were discussed by him. He also modified the continuation operators of Duffin, Kurówski and Abdullaev using q-difference theory and incorporated the convolution product with it. As against the classical case, the fundamental theorem of algebra does not hold good in discrete function theory. Isaacs, Terracini and Harman investigated the roots of discrete polynomials.

Later Zeilberger [127-133] introduced a few results such as discrete powers and entire functions in the set of Gaussian integers. Recently Subhash Kak [101] extended Duffin's theory of Hilbert transform to the realm of electronics. Mugler [91] also studied exponential functions. Velukutty [117-118] studied q-monodiffric and bianalytic functions. Mercy [86-88] extended the pseudoanalytic theory of Bers [14-15] in the light of Harman's work. Vijay Kumar [119-122] studied holometric space.

Khan [76], in his Doctoral thesis, introduced discrete analogue of q-Hypergeometric functions using the discrete power $z^{(n)}$ given by Harman. He [76,77] studied transformations and expansion formulae of these discrete hypergeometric functions. He [76,78] gave the concept of bibasic analytic functions on two unconnected bases p and q and introduced discrete bibasic Hypergeometric functions. For these discrete bibasic hypergeometric functions, Bahadur [12] obtained expansion formulae similar to those obtained by Khan [76] for discrete q-Hypergeometric functions. Harman [58] also studied discrete functions on a radial lattice.

3. DEFINITIONS AND NOTATIONS. For $|q| < 1$, let

$$[\alpha] = (1-q^{\alpha})/(1-q) \qquad \ldots\ldots (1.3.1)$$

$$[\alpha]_{n,q} = [\alpha]_n = [\alpha][\alpha+1]\ldots.[\alpha+n-1];[\alpha]_0 = 1 \qquad \ldots\ldots (1.3.2)$$

$$[n] \equiv [1]_{n,q} = (1-q)_n/(1-q)^n \qquad \ldots\ldots (1.3.3)$$

$$(q^{\alpha})_n \equiv (\alpha)_{n,q} = (1-q^{\alpha})(1-q^{\alpha+1})\ldots(1-q^{\alpha+n-1});\ (q^{\alpha})_0 = 1 \qquad \ldots\ldots (1.3.4)$$

$$(q^{(\alpha_r)})_n \equiv ((\alpha_r))_{n,q} = \prod_{j=1}^{n} (q^{\alpha_j})_n \qquad \ldots\ldots (1.3.5)$$

$$(q^{\alpha})_{-n} = q^{n(n+1-2\alpha)/2}\,(-1)^n/(q^{1-\alpha})_n \qquad \ldots\ldots (1.3.6)$$

$$(q^{\alpha})_{m+n} = (q^{\alpha})_m\,(q^{\alpha+m})_n \qquad \ldots\ldots (1.3.7)$$

$$(1+x)_n = (1+x)(1+qx)\ldots\ldots(1+q^{n-1}x);\ (1+x)_0 = 1 \ \ldots\ (1.3.8)$$

$$(1+x)_{\infty} = \lim_{n\to\infty} (1+x)_n \qquad \ldots\ldots (1.3.9)$$

$$\binom{n}{k}_q \equiv \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]}{[k]\ [n-k]} = \frac{(q)_n}{(q)_k\,(q)_{n-k}};\ \binom{n}{0}_q = 1 \qquad \ldots\ldots (1.3.10)$$

where m and n are non-negative integers.

A generalized q-Hypergeometric function is then defined as

$$ {}_r\phi_s^{(q)}[(a_r);(b_s);z] = {}_r\phi_s^{(q)} \begin{bmatrix} (a_r); & z \\ (b_s); & \end{bmatrix} = {}_r\phi_s \begin{bmatrix} q^{(a_r)} & ;\ z \\ q^{(b_s)} & ; \end{bmatrix} = \sum_{n=0}^{\infty} \frac{((a_r))_{n,q}\, z^n}{(1)_{n,q}\,((b_s))_{n,q}} \qquad \ldots\ldots (1.3.11)$$

where q is called the base of the ${}_r\phi_s$-function. When $|q| < 1$, the infinite series (1.3.11) is absolutely convergent for $|z| < 1$.

The ${}_r\phi_s$-function yields as very special cases the Weierstrass's and Jacobi's elliptic functions and reduces to an ordinary hypergometric function ${}_rF_s$ as $q \longrightarrow 1$.

Usually in the definition of (1.3.11), z is not a function of n, we may, however, take z to be a function of n also. As for instance a series of the type

$$\sum_{n=0}^{\infty} \frac{((a_r))_{n,q}\, z^n\, q^{\lambda n(n+1)/2}}{(1)_{n,q}\, ((b_s))_{n,q}} \qquad \ldots\ldots (1.3.12)$$

is denoted by ${}_r\phi_s^{(q)}\left[\begin{matrix}(a_r); & z\\ (b_s); & \end{matrix}\right]$ or simply by ${}_r\phi_s(z,\lambda)$.

The q-analogue of a binomial product, exponential function, Sin x, Cos x, gamma function and Bessel function are given below:

$$(x-y)_\alpha = x^\alpha \sum_{n=0}^{\infty} \left[\frac{1-(y/x)\ q^n}{1-(y/x)\ q^{\alpha+n}}\right] \qquad \ldots\ldots (1.3.13)$$

<u>Basic exponentials</u>

$$e_q(x) = \prod_{n=0}^{\infty} [1-xq^n]^{-1} = \sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \qquad \ldots\ldots (1.3.14)$$

$$E_q(x) = (1-x)_\infty = \sum_{r=0}^{\infty} \frac{(-1)^r \, q^{r(r-1)/2}}{(q)_r} x^r \quad \ldots\ldots (1.3.15)$$

Basic trigonometric functions

$$\mathrm{Sin}_q\, x = \frac{1}{2i}\,[e_q(ix)-e_q(-ix)] = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(q)_{2r+1}} \quad ..(1.3.16)$$

$$\mathrm{Cos}_q\, x = \tfrac{1}{2}\,[e_q(ix)+e_q(-ix)] = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(q)_{2r}} \quad \ldots\ldots (1.3.17)$$

$$\mathrm{Sin}_q\, x = \frac{1}{2i}\,[E_q(ix)-E_q(-ix)] = \sum_{r=0}^{\infty} (-1)^r \frac{q^{2r+1}}{(q)_{2r+1}} x^{2r+1} \quad \ldots\ldots (1.3.18)$$

$$\mathrm{Cos}_q\, x = \tfrac{1}{2}\,[E_q(ix)+E_q(-ix)] = \sum_{r=0}^{\infty} (-1)^r \frac{q^{r(2r-1)}}{(q)_{2r}} x^{2r} \quad \ldots\ldots (1.3.19)$$

where $|x| < 1$.

Basic gamma function

$$\Gamma_q(x) = (1-q)_{\alpha-1}/(1-q)^{\alpha-1}, \ (\alpha \neq 0,-1,-2,\ldots) \quad \ldots\ldots (1.3.20)$$

Basic Bessel functions

$${}_qJ_\alpha(x) = \frac{1}{(q)_\alpha} \left(\frac{x}{2}\right)^\alpha {}_0\Phi_1\left[\begin{matrix} —; & -x^2\, q^{\alpha+1/4} \\ q^{1+\alpha}; & q^2 \end{matrix}\right] \quad \ldots\ldots (1.3.21)$$

$$ {}_q j_\alpha(x) = \frac{1}{(q)_\alpha} \left(\frac{x}{2}\right)^\alpha {}_0\phi_1 \left[\begin{matrix} \text{—}\, ; & -x^2/4 \\ q^{1+\alpha}; & \end{matrix}\right] \qquad \ldots\ldots (1.3.22) $$

In order to formulate q-addition theorem we also need to define the following notations (see Hahn [47]):

Let

$$ f(x) = \sum_{r=0}^{\infty} a_r x^r \qquad \ldots\ldots (1.3.23) $$

be a power series in x. Then

$$ f([x-y]) = \sum_{r=0}^{\infty} a_r (x-y)_r \qquad \ldots\ldots (1.3.24) $$

and

$$ f\left(\frac{t}{[x-y]}\right) = \sum_{r=0}^{\infty} a_r \frac{t^r}{(x-y)_r} \qquad \ldots\ldots (1.3.25) $$

Still another q-concept of importance is that of q-integration. Jackson [74] in 1910 first introduced this concept as the inverse operator of the q-difference operator, viz.,

$$ D_{q,x}\, f(x) = D_q\, f(x) = \frac{f(x) - f(xq)}{(1-q)\,x}, \ |q| \neq 1 \qquad \ldots\ldots (1.3.26) $$

The q-integration operator was denoted by him as

$$D_{q,x}^{-1} f(x) = \frac{1}{(1-q)} \int f(x) d(x,q) \qquad \ldots\ldots (1.3.27)$$

It is apparent that if f is differentiable then

$$\lim_{q \to 1} D_{q,x} f(x) = \frac{df}{dx} \qquad \ldots\ldots (1.3.28)$$

It was not until 1949, that an extensive and rigorous study of q-integration was made by Hahn [47] and later in 1951, by Jackson [75] who studied the fundamental properties of the inverse operation $D_{q,x}^{-1} f(x)$ and showed that, under certain conditions, the q-integral tends to the Riemann integral as $q \longrightarrow 1$.

The definite q-integrals are defined by

$$\int_0^x D_{q,x} f(x) \; d(x;q) = f(x) - f(0) \qquad \ldots\ldots (1.3.29)$$

$$\int_x^\infty D_{q,x} f(x) \; d(x;q) = f(\infty) - f(x) \qquad \ldots\ldots (1.3.30)$$

whence

$$\int_x^\infty f(t) d(t;q) = \int_0^b f(t) d(t;q) - \int_0^a f(t) d(t;q) \qquad \ldots\ldots (1.3.31)$$

Correspondingly, the q-integrals can be defined by the relations

$$\int_0^x f(t)\,d(t;q) = x(1-q) \sum_{n=0}^{\infty} q^n f(x q^n) \qquad \ldots\ldots (1.3.32)$$

$$\int_x^\infty f(t)\,d(t;q) = x(1-q) \sum_{n=1}^{\infty} q^{-n} f(x q^{-n}) \qquad \ldots\ldots (1.3.33)$$

$$\int_0^\infty f(t)\,d(t;q) = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) \qquad \ldots\ldots (1.3.34)$$

and so the convergence of the q-integral is determined by the convergence of the corresponding q-sum.

Throughout this thesis a notation of the type (1.2.3) will mean 3rd equation of the 2nd article of Ist Chapter. Also (1.2.3-5) will mean (1.2.3), (1.2.4) and (1.2.5).

4. MONODIFFRIC FUNCTIONS. In 1941, Isaacs [69] modified the concept of monogeneity and developed a theory of complex-valued functions defined at the points of the complex plane whose coordinates are integers. These points form a lattice which breaks up the plane into unit squares. His theory of discrete analytic functions is based on the following definition of analyticity:

$$f(z+1) - f(z) = \frac{f(z+i) - f(z)}{i}, \quad i = \sqrt{-1} \qquad \ldots\ldots (1.4.1)$$

Using this concept of a difference quotient instead of a derivative, Isaacs constructed a theory for functions defined only on the set of Gaussian integers (points of the form $m+in$; m,n integers). This was the first attempt to devise a 'discrete analytic function theory'.

Isaacs [69,70] in fact defined two types of discrete functions. Those satisfying (1.4.1) he termed 'monodiffric functions of the first kind'. He preferred the definition (1.4.1) to

$$f(z+1) - f(z-1) = \frac{f(z+i) - f(z-i)}{i} \qquad \ldots\ldots \ (1.4.2)$$

which he also considered. Functions satisfying (1.4.2) he called 'monodiffric functions of the second kind'. In each case the domain of definition of the function was the set of square lattice points - the Gaussian integers. He introduced concepts and theories for discrete contour integrals, residues, powers, polynomials and a convolution which served as an analogue for multiplication of discrete functions, provided one of them was a polynomial Chapter II is a survey of the theory of monodiffric functions developed by Isaacs [69,70].

5. DISCRETE q-DIFFERENCE FUNCTIONS. Discrete function theory is concerned with a study of functions defined only at certain lattice points in the complex plane. Using ordinary-difference operators instead of derivatives, Isaacs [69] in 1941 introduced the concept of a discrete analytic functions defined on the set of Gaussian integers. The subject was given impetus by a number of mathematicians but their works were mainly in the set of Gaussian integers. The theory of geometric difference or q-difference functions are a generalization of ordinary differences and constitute an important branch of finite difference theory. In contrast to the arithmetic spacing of the Gaussian integers (fixed distance between two lattice points) Harman [56,57] developed a theory for functions defined on a lattice with geometric spacing (points of the form $\{(\pm q^m x', \pm q^n y'); q$ fixed, m, n integers$\}$). A discrete analytic function theory is evolved which applies to q-difference functions. The partial q-difference operators θ_x and θ_y are defined as follows:

$$\theta_x [f(z)] = \frac{f(z) - f(qx,y)}{(1-q)x} \qquad \text{...... (1.5.1)}$$

$$\theta_y [f(z)] = \frac{f(z) - f(x,qy)}{(1-q)iy} \qquad \text{...... (1.5.2)}$$

where f is a discrete function. The two operators involve a basic triad of points denoted by $T(z) = \{(x,y),(qx,y),(x,qy)\}$. If D is a discrete domain, then a discrete function f is said to be q-analytic at $z \in D$ if

$$\theta_x f(z) = \theta_y f(z) \qquad \ldots\ldots (1.5.3)$$

If in addition this equality holds for every z D such that $T(z) \subseteq D$ then f is said to be q-analytic in D.

Thus using q-difference operators instead of derivatives to define a discrete analytic function, Harman [56,57] established analogues for contour integration, Cauchy integral theorems, culminating in an analogue of Cauchy's integral formula. He also devised a discrete analytic continuation operator C defined by

$$C_y = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j \theta_x^j \qquad \ldots\ldots (1.5.4)$$

so that

$$f(z) = C_y [f(x,0) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j \theta_x^j [f(x,0)] \ldots (1.5.5)$$

similarly,

$$C_x\,[f(0,y)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j}\, x^j\, \theta_y^j\,[f(0,y)], \qquad \ldots\ldots (1.5.6)$$

Thus the operator C enables functions defined on the axes to be continued into the complex plane as q-analytic functions. This process (infact an analogue of Taylor's theorem) is of fundamental importance to the development of the subsequent theory.

An important problem in discrete analytic function theory is the construction of an analogue for multiplication. An operator * is sought such that if f,g are two discrete analytic functions, then f*g is also discrete analytic. At the same time the operator * should retain many of the desirable algebraic properties of the classical operation of multiplication. Harman defined the convolution operator * as follows:

$$(f*g)(z) = C_y[f(x,0)g(x,0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j}\,(iy)^j\, \theta_x^j\,[f(x,0)g(x,0)] \ \ldots\ldots (1.5.7)$$

Harman's operator * preserves most of the important properties of the classical operation of multiplication.

He also used the continuation operator C to derive q-analogue of the function z^n; n a non-negative integer. He also examined the analogue of the function $(z-z_0)^\alpha$ where α is an arbitrary complex number. This appears to be the first such analogue in discrete analytic function theory, and of course includes the case of negative powers. He further examined discrete polynomials in the q-analytic theory and used the convolution operator * to derive a factorization result.

Chapter III is a survey of q-analytic function theory developed by Harman [52-57].

6. DISCRETE q-HYPERGEOMETRIC FUNCTIONS. Harman [52] defined, for a non-negative integer n, a q-analytic function $z^{(n)}$ to denote the discrete analogue of z^n, satisfying the following conditions:

$$\left.\begin{aligned} D_q\,[z^{(n)}] &= [n]\, z^{(n-1)} \\ z^{(0)} &= 1 \\ 0^{(n)} &= 0,\ n > 1 \end{aligned}\right\} \qquad \ldots\ldots\ (1.6.1)$$

The operator C_y when applied to the real function x^n, yields $z^{(n)}$. In fact, Harman defined $z^{(n)}$ by

$$z^{(n)} = C_y(x^n);\ n \text{ a non-negative integer}$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D^j_{q,x} (x^n) \qquad \ldots\ldots (1.6.2)$$

which on simplification, yields

$$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j}_q x^{n-j} (iy)^j \qquad \ldots\ldots (1.6.3)$$

Using Harman's discrete analogue $z^{(n)}$ for the classical function z^n as given by (1.6.3), Khan [76] defined a discrete analogue of the q-Hypergeometric function ${}_r\phi_s^{(q)}[(a_r);(b_s);z]$ by means of the following relation:

$${}_rM_s[(a_r);(b_s);q,z] = C_y\left\{ {}_r\phi_s^{(q)}[(a_r);(b_s);x]\right\}$$

$$= \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n\, z^{(n)}}{(q)_n\, (q^{(b_s)})_n} \qquad \ldots\ldots (1.6.4)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{(a_r)})_{n+k}\, x^n (iy)^k}{(q)_n\, (q)_k\, (q^{(b_s)})_{n+k}} \qquad \ldots\ldots (1.6.5)$$

Besides proving it to be q-analytic he also discussed its elementary properties and obtained certain elegant transformations and expansion formulae but due to paucity of space these transformations and extansions formulae have been omitted in chapter IV which only gives a glimpse of a new direction for research works to be carried further.

7. DISCRETE FUNCTIONS ON A RADIAL LATTICE. In Chapter V a discrete model for analytic functions constructed by Harman [58] by using lattice points of the complex plane arranged in radial form is described. The discrete analytic functions are defined as solutions of a finite difference approximation to the polar Cauchy-Riemann equations. The resulting discrete power $z^{(n)}$ (an analogue of z^n) has as a simple algebraic form (a direct analogue of $\rho^n \exp in\theta$) and has some surprising properties. For example every discrete polynomial $\sum_{n=0}^{m} a_n z^{(n)}$ has a factorization in terms of the zeros of its classical counterpart $\sum_{n=0}^{m} a_n z^n$ and every discrete entire function has a power series representation $\Sigma\, a_n\, z^{(n)}$.

II CHAPTER

MONODIFFRIC FUNCTION THEORY

1. INTRODUCTION. Although the calculus of finite differences has flourished for well over a century, there has never been a full development of its extension in the realm of the complex variable. In other words there has been no attempt made till 1941 to construct a theory of functions of a complex variable which springs entirely and basically from the concept of the difference quotient instead of the derivative.

In 1941, Isaacs [69] singled out a special class of function which have a certain uniqueness of difference quotient instead of the unique derivative in detail they were the functions whose difference quotient is the same when taken in the purely real and purely imaginary directions. He called these functions monodiffric. For this he,considered the aggregate of all functions of all complex variable $z = x+iy$ which admit of partial derivatives; i.e. pairs of real functions of two real variables such that the four partial derivatives exist. Such he called, with Kasner, polygenic functions of all these functions a special subclass is usually singled out for study-those which leave a derivative at each point independent of the direction of approach of the increamented point; that is those $f(z)$ for which

$$\lim_{\rho \to 0} \frac{f(z+\rho e^{i\theta}) - f(z)}{\rho e^{i\theta}}$$

is independent of θ. These functions, with Cauchy, he called monogenic. The study of the function properties that ensue from this restriction is so fruitfull that it constitutes a major branch of mathematics. Isaacs pointed out that monodiffric functions form a class about as extensive as the monogenic functions and they are not quite but nearly as rich in properties or it was harder to find those properties.

The kernel of the classic function theory are the theorems dealing with contour integrals and residues. He found analogous theorems in his work and expected them to provide a method of great power. In as much as his theory can be regarded as applying only to the lattice of gaussian integers results followed which were arithmatic in character. He hoped that if such ideas could be sufficiently developed they might yield a near and revolutionary connection between number theory and analysis.

2. MONODIFFRIC FUNCTIONS. The most obvious method of writing a polygenic function is

$$f(z) = u(x,y) + iv(x,y)$$

where u,v,x,y are real. It is naturally desirable to find a method similar to the customary way of writing monogenic functions as explicit functions of z. This is done by regarding

$$z = x+iy$$

$$\bar{z} = x-iy$$

as linear transformation and performing its inverse

$$x = \frac{1}{2}(z+\bar{z})$$

$$y = \frac{1}{2i}(z-\bar{z})$$

on f. The resulting $F(z,\bar{z})$ written explicitly in its two arrangements is call the conjugate form off.

$D_z F$ and $D_{\bar{z}} F$ are the formal partial derivatives of F with respect to z and $\bar{z}$. It is easy to show

$$D_z = \frac{1}{2}(D_x - iD_y)$$

$$D_{\bar{z}} = \frac{1}{2}(D_x + iD_y) \qquad \ldots\ldots (2.2.1)$$

The directional derivative operator is

$$D_\theta = \cos\theta\, D_x + \sin\theta\, D_y \qquad \ldots\ldots (2.2.2)$$

In conjugate form it is

$$D_\theta = e^{i\theta} D_z + e^{-i\theta} D_{\bar{z}} \qquad \ldots\ldots (2.2.3)$$

obtained by substituting (2.2.1) into (2.2.2) $\frac{df}{dz}$ taken in the θ direction is

$$[D_z + e^{-2i\theta} D_{\bar{z}}]F \qquad \ldots\ldots (2.2.4)$$

which is independent of θ if and only if $D_{\bar{z}} F = 0$, that is if F is free of $\bar{z}$. In the non-conjugate form this condition is the Cauchy-Riemann equations. It is clear that for a fixed z and varying θ the points (2.2.4) lie in a circle with centre at $D_z F$ and radius $[D_{\bar{z}} F]$, which is often called the Kasner circle.

Keeping these in mind Isaacs [69] modified the concept of monogeneity and introduced the notion of a monodiffric functions. For this he was faced with two procedures:

(i) He defined

$$\Delta_x f(z) = f(z+1) - f(z)$$

$$\Delta_y f(z) = f(z+i) - f(z)$$

Functions for which

$$\Delta_x f(z) = \frac{1}{i} \Delta_y f(z) \qquad \ldots\ldots (2.2.5)$$

hold, he termed them as 'monodiffric functions of the first kind'.

(ii) He also defined

$$\Delta_x f(z) = \frac{f(z+1) - f(z-1)}{2}$$

$$\Delta_y f(z) = \frac{f(z+i) - f(z-i)}{2}$$

Functions for which (2.2.5) holds with the new definitions i.e. one that satisfies

$$f(z+1) - f(z-1) = \frac{f(z+i) - f(z-1)}{i} \qquad \ldots\ldots (2.2.6)$$

he called 'monodiffric functions of the second kind'.

He kept the same notation as many theorems hold for both types of functions. In case of any distinction he used the number (i) and (ii) to distinguish the two cases.

For monodiffric functions he denoted by $\Delta f(z)$ the common value of $\Delta_x f(z)$ and $\frac{1}{i} \Delta_y f(z)$.

Although functions (ii) are more symmetric functions (i) proved to be more useful this is because the defining equation (ii) is really a difference equation of second order.

The definition (i) involves three points so related that if one of them is a, the other two are $a+1$ and $a+i$. The definition (ii) involves four points $a+1$, $a-1$, $a+i$, $a-i$. Such sets of points he called as associated points. For both types of function he proved the following theorems:

THEOREM 2.2.1. If $f(z)$ is monodiffric so is $\Delta f(z)$.

THEOREM 2.2.2. If $f(z)$ and $g(z)$ are monodiffric so is $af+bg$ and $\Delta(af+bg) = a\Delta f + b\Delta g$; i.e. Δ is a linear operator.

THEOREM 2.2.3. Let $\{f_n(z)\}$ $(n = 1,2,\ldots)$ be a sequence of monodiffric functions which approach a limit. Then the limit is monodiffric; $\lim \Delta f_n(z)$ exists and equals $\Delta \lim f_n(z)$.

As an alternative to his procedure Isaacs also dealt with differences with an arbitrary complex span $h \neq 0$; when $f(z)$ satisfies

$$\frac{1}{h}[f(z+h)-f(z)] = \frac{1}{ih}[f(z+ih)-f(z)] = {}_h\Delta f(z) \qquad \ldots\ldots (2.2.7)$$

he called it h-monodiffric.

THEOREM 2.2.4. If $f(z)$ is monodiffric $f(\frac{z}{h})$ is h-monodiffric.

By seeking the conditions that must be satisfied separatel by the real and imaginary parts of a monodiffric function. Isaacs found that his work paralleled closely the common elementary function theory.

Eqn. (2.2.5) can be written as

$$(\triangle_x + i\triangle_y)(u+iv) = 0$$

$$\begin{aligned} \triangle_x u &= \triangle_y v \\ \triangle_y u &= -\triangle_x v \end{aligned} \qquad \text{......} \quad (2.2.8)$$

which immediately gives

$$(\triangle_x^2 + \triangle_y^2)u = 0 \qquad \text{......} \quad (2.2.9)$$

and similarly for v. Functions satisfying (2.2.9) he termed as diharmonic.

3. SUMMATION. Isaac also established procedure analogous to contour integration and, in a sense, an inverse of the operation $\triangle$.

By a path on a basic lattice he meant a polygonal line which is the union of sides of basic squares (i.e. a lattice square where by a lattice he meant the set of intersections of two perpendicular families of equidistant parallel lines, both families having the same spacing).

By considering P as a path joining the two lattice points a and b and f to be a polygenic function, he defined the path sum of f over P as

$$\sum_{a(P)}^{b} f(z) \triangle z \qquad \ldots\ldots (2.3.1)$$

and proved the following theorem:

THEOREM 2.3.1. The path sum of a monodiffric function over a closed path is zero, and if the path sum of f over all closed paths is zero, f is monodiffric.

Thus for monodiffric functions summation is independent of the path which enabled him to define what he called the indefinite sum of f. Choosing any fixed basic square and calling it Q, he pointed out that every point z is congruent to a point of Q which he called $q(z)$. Then the indefinite sum of $f(z)$ is

$$Sf(z) = \sum_{q(z)}^{z} f(\alpha)\triangle\alpha .$$

S is the converse of $\triangle$ as shown by his following theorem:

THEOREM 2.3.2. $Sf(z)$ is a monodiffric function of z and $\triangle[Sf(z)] = f(z)$.

He called $w(z)$ a basic periodic function if it is doubly periodic function with a basic square as period parallelogram.

THEOREM 2.3.3. $Sf(z)$ is determined uniquely to within an arbitrary additive basic period function

Let $S_1f(z)$ and $S_2f(z)$ be two indefinite sums of f and

$$w(z) = S_1f(z) - S_2f(z).$$

Then

$$\triangle w = \triangle S_1 f - \triangle S_2 f.$$

As $\triangle w = 0$, we have

(i) $w(z+1) - w(z) = 0$

$w(z+i) - w(z) = 0$

which shows w is basic periodic, (ii) similarly.

4. MONODIFFRIC POLYNOMIALS. In this section we will encounter the first explicit exhibition of monodiffric functions.

If $f(z)$ is a polygenic polynomial, by its degree we mean the total degree of $F(z,\overline{z})$ in z and $\overline{z}$. The following theorems hold:

THEOREM 2.4.1. If $f(z)$ is a monodiffric polynomial of degree n, then it is of the form $kz^n + G(z,\overline{z})$ where $k \neq 0$ and the degree of $G < n$.

THEOREM 2.4.2. If f is a monodiffric polynomial of degree n, Δf is of degree $n-1$.

THEOREM 2.4.3. Let $f(z)$ be a monodiffric polynomial whose degree is known not to exceed $n > 0$. Let $f(0) = 0$, $\Delta f(0) = 0$, ..., $\Delta^n f(0) = 0$. Then $f(z) \equiv 0$.

Isaacs [69] constructed a sequence of monodiffric polynomials which he designated by $z^{(0)} = 1$, $z^{(1)}$, $z^{(2)}$,........ He called them pseudo powers of z which have the following properties:

1. $\triangle z^{(n)} = n\, z^{(n-1)}$

2. $0^{(n)} = 0$ for $n > 0$.

He pointed out that if one knows $z^{(n)}$ then $z^{(n+1)}$ must be $(n+1) S\, z^{(n)}$. By theorem 2.3.3 and property 2 this is unique to within a basic periodic function which vanishes at zero. Of the functions so obtainable at most one can be a polynomial. The uniqueness of $z^{(n)}$ follows by induction.

THEOREM 2.4.4. Let $f(z)$ be a monodiffric polynomial of degree n. Then

$$f(z) = \sum_{j=0}^{n} \frac{\triangle^j f(0)}{j!} z^{(j)} \qquad \ldots\ldots (2.4.1)$$

This establishes an analogue of Taylor's theorem. An important special case is the binomial theorem for positive integral n.

$$(a+b)^{(n)} = \sum_j \binom{n}{j} a^{(j)}\, b^{(n-j)} \qquad \ldots\ldots (2.4.2)$$

which follows on applying Theorem 2.4.4 to $f(z) = (z+b)^{(n)}$ and taking $z = a$.

In particular

$$z^{(n)} = (x+iy)^{(n)} = \sum_j \binom{n}{j} x^{(j)} (iy)^{(n-j)} \qquad \ldots\ldots (2.4.3)$$

where $x^{(j)}$ is the value assumed by $z^{(j)}$ on the real axis. In view of two kinds of monodiffric functions it must satisfy

1. (i) $(x+1)^{(n)} - x^{(n)} = n\, x^{(n-1)}$

   (ii) $$\frac{(x+1)^{(n)} - (x-1)^{(n)}}{2} = n\, x^{(n-1)}$$

2. $x^{(0)} = 1,\ 0^{(n)} = 0 \qquad (n = 1,2,\ldots\ldots)$

If the $x^{(n)}$ exist they are unique.

THEOREM 2.4.5. (i) $x^{(n)} = x\,(x-1)\ldots\ldots.(x-(n-1))$

(ii) for odd n

$$x^{(n)} = [x+(n-2)]\ldots[x+3][x+1]x[x-1][x-3]\ldots[x-(n-2)];$$

for even n

$$x^{(n)} = [x+(n-2)]\ldots[x+4][x+2]x^2[x-2][x-4]\ldots[x-(x-2)].$$

On the imaginary axis $z^{(n)}$ assumes the value $(iy)^{(n)}$ which satisfies

1. (i) $$\frac{(i(y+1))^{(n)} - (iy)^{(n)}}{i} = n(iy)^{(n-1)}$$

   (ii) $$\frac{(i(y+1))^{(n)} - (i(y-1))^{(n)}}{2} = n(iy)^{(n-1)}$$

2. $(iy)^{(0)} = 1, \quad (i0)^{(n)} = 0. \qquad (n = 1,2,......)$

He then exhibited $z^{(n)}$ as follows:

THEOREM 2.4.6.

$$z^{(n)} = \sum_j \binom{n}{j} i^{n-j} x^{(j)} y^{(n-j)} \qquad ...... (2.4.4)$$

where the last two factors are the real function of Theorem 2.4.5.

Thus he completely exhibited all monodiffric polynomials.

III CHAPTER

DISCRETE GEOMETRIC FUNCTION THEORY

1. INTRODUCTION. Geometric or q-difference functions are a generalization of ordinary differences and constitute an important branch of finite difference theory. In this chapter, an extension of q-difference theory into the realm of discrete analytic functions is discussed which was originated in 1972 by C.J. Harman [52]. The concepts of q-difference and q-integration operators, defined and developed by Jackson [71,75] Hahn [47] and others, are extended into the complex plane comprising lattice points with unequal, logarithmic spacing. A class of functions (q-analytic) defined on this set is examined, a discrete contour integral results, and, using this, analogues are obtained for Cauchy integral theorems and his integral formula.

Results are outlined for a discrete analogue of analytic continuation and a powerful method is devised for the continuation (into the discrete plane) of functions defined on the axes. A convolution operator ensue which is analogous to multiplication of functions in the classical continuous theory. It is shown to have significant advantages over existing operators in the discrete analytic theory associated with ordinary-differences.

Also, in this chapter discrete analytic theory of geometric difference (or q-difference) functions is extended to a consideration of finite series. Appropriate continuation and convolution operators are utilized to obtain the function $z^{(n)}$ (the discrete analogue of the function z^n) and other elementary functions. The convenient form of $z^{(n)}$ leads to certain fundamental results for the representation of geometric functions in terms of power series. In particular, suitable discrete exponential, trignometric and power functions are defined and their properties examined. Finally it is shown that discrete analogues of Taylor's series apply in the theory.

2. THE LATTICE. In the theory of finite differences it is usual to define q-difference functions on a set of points of the form $\{q^n x;\ n \in Z,\}$ the set of integers . In order to construct a discrete analytic theory for q-difference functions of a complex variable, Harman [56] felt it necessary to introduce a suitable lattice. In fact he defined such functions on a geometric lattice.

For this he defined the discrete plane Q^1, with respect to some fixed point $z^1 = x^1 + iy^1 = (x^1, y^1)$ in the first quadrant, by the set of lattice points $Q^1 = \{(q^m x^1,\ q^n y^1);\ m, n \in Z\}$. Figure 1 shows a portion of Q^1 and demonstrates that the discrete plane is represented by a rectangular set of lattice points. The distance between adjacent points is not fixed, in contrast with the standard theories of discrete analytic functions. In the figure some examples of points are given; where

$$z^1 = (x^1, y^1), z_1 = (qx^1, y^1), z_2 = (qx^1, qy^1), z_3 = (x^1, qy^1)$$

$$z_4 = (q^{-1}x^1, qy^1), z_5 = (q^{-1}x^1, y^1), z_6 = (q^{-1}x^1, q^{-1}y^1)$$

$$z_7 = (x^1, q^{-1}y^1), z_8 = (qx^1, q^{-1}y^1).$$

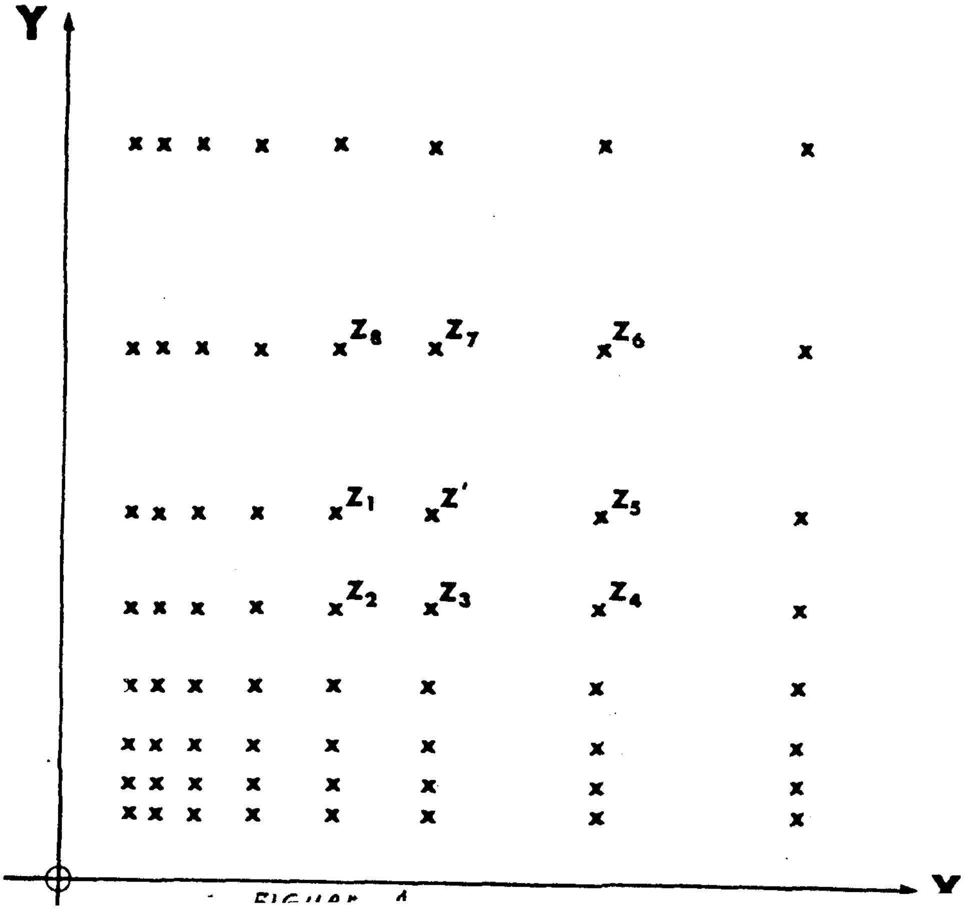

FIGURE 1

Two lattice points z_i, $z_{i+1} \in Q^1$ are said to be adjacent if z_{i+1} is one of (qx_i, y_i), $(q^{-1}x_i, y_i)$, (x_i, qy_i) or $(x_i, q^{-1}y_i)$. A discrete curve C in Q^1, connecting z_o to z_n is denoted by ordered sequence

$$C = \langle z_o, z_1 \ldots\ldots z_n \rangle,$$

where z_i, z_{i+1}; $i = 0, 1, \ldots\ldots n-1$, are adjacent points of Q^1. If the points are distinct $(z_i \neq z_j;\ i \neq j)$ then the discrete curve C is said to be simple. A discrete closed curve C in Q^1 is given by a sequences $\langle z_o, z_1, \ldots\ldots z_n \rangle$ where $\langle z_o, z_1, \ldots\ldots z_{n-1} \rangle$ is simple and $z_o = z_n$. If the continuous closed curve, formed by joining adjacent points of discrete closed curve C, is denoted by $\bar{C}$, then $\bar{C}$ encloses certain points of Q^1 denoted by Int(C). A finite discrete domain D is defined by $D = \{z;\ z \quad C \cup \mathrm{Int}(C)\}$, and in general, a discrete domain D is defined as a union of finite discrete domains.

Defining a basic set with respect to $z \in Q^1$ as $S(z) = \{(x,y),\ (qx,y),\ (qx,qy),\ (x,qy)\}$, then it follows that a finite discrete domain D is a union of basic sets.

3. q-ANALYTIC FUNCTIONS. Functions defined on the points of

a discrete domain D are said to be discrete functions. The partial q-difference operators Θx, Θy are defined as follows:

$$\Theta_x[f(z)] = \frac{f(z) - f(qx,y)}{(1-q)x}$$

$$\Theta_y[f(z)] = \frac{f(z) - f(x,qy)}{(i-q)iy} \qquad \ldots\ldots (3.3.1)$$

where f is a discrete function. The two operators involve a basic triad of points denoted by - $T(z)=\{(x,y),(qx,y)(x,qy)\}$. If D is a discrete domain then a discrete function f is said to be q-analytic at $z \in D$ if

$$\Theta_x f(z) = \Theta_y f(z) \qquad \ldots\ldots (3.3.2)$$

If in addition this equality holds for every $z \in D$ such that $T(z) \subseteq D$ then f is said to be q-analytic in D. For simplicity, if a function is q-analytic, then a common operator Θ can be used where $\Theta = \Theta_x = \Theta_y$.

4. PROPERTIES OF q-ANALYTIC FUNCTIONS. The operator L is to be defined by

$$Lf(z) = \bar{z}f(z) - xf(x,qy) + iyf(qx,y). \qquad \ldots\ldots (3.4.1)$$

From (3.3.2) f is q-analytic in a discrete domain D if and only if

$$Lf(z) = 0 \qquad \text{...... (3.4.2)}$$

for every $z \in D$ with $T(z) \subseteq D$.

Now a discrete domain D is a union of basic sets S.

If

$$D = \bigcup_{i=1}^{N} S(z_i)$$

then the domain D_q is defined as the set of points z_i which generate the basic sets, i.e. $D_q = \{z_i, z_i \in S(z_i),\ i=1,2,\ldots N\}$ where N can of course be infinite.

The following results demonstrate that the class of q-analytic functions exhibit properties similiar to differentiable functions.

THEOREM 3.4.1. If a discrete function f is q-analytic in D, then Θf is q-analytic in D_q.

From the definition it is readily seen that Θ is a linear operator and hence that the following theorem and corollary hold.

THEOREM 3.4.2. The class of q-analytic functions in D forms a vector space over the field of complex numbers.

COROLLARY 3.4.1. The sum of a finite number of q-analytic functions is q-analytic.

The following theorem and corollary also follow readily from the preceding definitions.

THEOREM 3.4.3. If $\{f_n\}$ is a pointwise convergent sequence of q-analytic functions in D with limit f, then

(i) f is q-analytic in D, and

(ii) $\lim_{n \to \infty} \Theta f_n(z) = \Theta f(z)$; $z \in D$.

COROLLARY 3.4.2. If

$$f = \sum_{j=0}^{\infty} g_j$$

is a convergent series of discrete functions g_j which are q-analytic in D, then

(i) f is q-analytic in D, and

(ii) $\Theta f = \sum_{j=0}^{\infty} \Theta g_j$ in D.

Analogues for the Cauchy-Rieman conditions and Laplace's equation can be obtained. If $f(z) = u(z) + iv(z)$ is a q-analytic function such that u,v are real, then

$$\Theta_x u(z) = i\Theta_y v(z)$$

$$\Theta_x v(z) = -i\Theta_y u(z)$$

and $[\Theta_x^2 + (i\Theta_y)^2]\, u(z) = 0$, with a similar result for $v(z)$.

5. THE DISCRETE LINE INTEGRAL. The concept of q-integration, originated by Jackson [71,75] is now extended into the complex plane.

If z_j and z_{j+1} are two adjacent points of (qx_j,y_j), (x_j,qy_j), $(q^{-1}x_j,y_j)$ or $(x_j,q^{-1}y_j)$. The discrete line integral from z_j to z_{j+1} of a discrete function f, is defined by

$$\int_{z_j}^{z_{j+1}} f(\xi)\,d(q,\xi) = \begin{cases} (z_{j+1}-z_j)f(z_j);\, z_{j+1}=(qx_j,y_j)\text{ or }(x_j,qy_j) \\ \\ (z_{j+1}-z_j)f(z_{j+1});\, z_{j+1}=(q^{-1}x_j,y_j)\text{ or }(x_j,q^{-1}y_j) \end{cases} \quad \text{...... (3.5.1)}$$

In general if $C = \langle z_o,z_1,\ldots\ldots z_n\rangle$ is a discrete curve in D, then the discrete line integral from z_o to z_n along C is defined as

$$\int_{z_o}^{z_n} f(\xi)d(q,\xi) = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} f(\xi)d(q,\xi) \qquad \ldots\ldots (3.5.2)$$

For convenience, the discrete line integral around a discrete curve C will also be denoted as

$$\underset{c}{S} f(\xi)d(q,\xi).$$

The following elementary properties follow readily from the definition of the discrete line integral

(i) Let $C_1 = \langle z_o,z_1,\ldots\ldots z_m\rangle$ and $C_2 = \langle z_m,z_{m+1}\ldots\ldots z_n\rangle$ be two discrete curve in D. If f is a discrete function defined on D, then

$$\underset{C_1}{S} f(\xi)d(q,\xi)+ \underset{C_2}{S} f(\xi)d(q,\xi) = \underset{C_1+C_2}{S} f(\xi)d(q,\xi)$$

where $C_1+C_2 = \langle z_o,z_1,\ldots\ldots,z_m, z_{m+1},\ldots\ldots,z_n\rangle$

(ii) If $C = \langle z_o,z_1,\ldots\ldots,z_n\rangle$ then $-C$ denotes the sequence in the reverse order $\langle z_n,z_{n-1},\ldots\ldots,z_o\rangle$ and

$$\underset{C}{S} f(\xi)d(q,\xi) = - \underset{-C}{S} f(\xi)d(q,\xi).$$

(iii) If α denotes a scalar constant, then

$$\underset{C}{S}\ \alpha f(\xi)d(q,\xi) = \alpha \underset{C}{S} f(\xi)d(q,\xi)$$

(iv) If f, g are two discrete functions, then

$$\underset{C}{S}(f+g)(\xi)d(q,\xi) = \underset{C}{S} f(\xi)d(q,\xi) + \underset{C}{S} g(\xi)d(q,\xi)$$

(v) If $C = \langle z_o, z_1, \ldots\ldots, z_n \rangle$, then

$$\left| \underset{C}{S} f(\xi)d(q,\xi) \right| \leq M\ell$$

where

$$M = \max_{\xi \in C} |f(\xi)|$$

and ℓ denotes the curve length

$$\sum_{j=0}^{n-1} |z_{j+1} - z_j| .$$

(vi) If the series $\sum_{j=0}^{\infty} g_j(\xi)$

of discrete functions g_j converges uniformly for all points on a discrete curve C, then the series may be integrated term by term along the curve i.e.

$$\underset{C}{S} \sum_{j=0}^{\infty} g_j(z)d(q,z) = \sum_{j=0}^{\infty} \underset{C}{S}\ g_j(z)d(q,z).$$

(vii) Let $C_1 = \langle z_o, z_1, \ldots\ldots, z_n \rangle$, $C_2 = \langle \omega_o, \omega_1, \ldots\ldots, \omega_m \rangle$ be to discrete curves in Q^1. If $f(z,\omega)$ is a discrete function defined for $z \in C_1$, $\omega \in C_2$, then

$$\underset{C_2}{S}\left[\underset{C_1}{S} f(z,\omega)\, d(q,z)\right] d(q,\omega) = \underset{C_1}{S}\left[\underset{C_2}{S} f(z,\omega)\, d(q,\omega)\right] d(q,z).$$

6. DISCRETE INTEGRATION OF q-ANALYTIC FUNCTIONS. The discrete line integral defined in (3.5.2) is related to the monodiffric integral defined by Isaacs [69] and as such as similar properties. The following theorems and corrolaries are q-analogues of Isaac's result and will only be stated.

THEOREM 3.6.1. (Analogue of Cauchy's and Morera's theorem) A discrete function f is q-analytic in D if and only if the discrete integral around every discrete closed curve in D is zero.

The discrete indefinite integral is defined as

$$F(z) = \underset{a}{\overset{z}{S}} f(\xi)\, d(q,\xi)$$

where a, z belong to some discrete domain D, a being fixed.

THEOREM 3.6.2. (The fundamental theorem) If f is q-analytic in D, then $F(z)$ is independent of the discrete curve from a to z in D.

COROLLARIES 3.6.1. If f is q-analytic in D and $C = \langle z_o, z_1, \ldots\ldots, z_n \rangle$ is a discrete curve in D, then

$$\underset{C}{S}\, f(\xi)\, d(q,\xi) = F(z_n) - F(z_o)$$

(II) If f is q-analytic in D, then F is q-analytic in D and $\Theta F = f$.

(iii) If f is q-analytic and $C = \langle z_o, z_1, \ldots\ldots, z_n \rangle$ in D, then

$$\overset{z_n}{\underset{z_o}{S}}\, f(\xi)\, d(q,\xi) = f(z_n) - f(z_o)$$

(iv) If F_1, F_2 are given by

$$F_1(z) = \overset{z}{\underset{a_1}{S}}\, f(\xi)\, d(q,\xi), \quad F_2(z) = \overset{z}{\underset{a_2}{S}}\, f(\xi)\, d(q,\xi),$$

then $F_1(z) = F_2(z) + W(z)$, where W is an arbitrary function, q-periodic in each of its components; i.e. $W(z) = W(qx,y) = W(x,qy)$

7. AN ANALOGUE OF CAUCHY'S INTEGRAL FORMULA. In order to develop a discrete analogue of Cauchy's integral formula it is convenient to introduce the concept of a p-analytic function

where $p = q^{-1}$. The ϕx, ϕy are defined by

$$\phi_x f(z) = \frac{f(z) - f(px,y)}{(1-p)x}$$

$$\phi_y f(z) = \frac{f(z) - f(x,py)}{(1-p)iy}$$

and a discrete function f, defined on Q^1, is said to be p-analytic at z if $\phi x f(z) = \phi y f(z)$. This equation is equivalent to the relation $B[g(z)] = 0$, where the operator B is defined by

$$Bg(z) = \bar{z}g(z) - xg(x,py) + iyg(px,y) \qquad \ldots\ldots (3.7.1)$$

The following definition of a conjoint line integral is the q-function analogue of the one introduced by Isaacs [69]. If $C = \langle z_o, z_1, \ldots\ldots, z_n \rangle$ is a discrete curve in D, and if f and g are two discrete functions, then the conjoint line integral along C is defined as

$$\mathop{S}_{z_o}^{z_n} (f \oplus g)(\xi)d(q,\xi) = \sum_{j=0}^{n-1} \mathop{S}_{z_j}^{z_{j+1}} (f \oplus g)(\xi)d(q,\xi)$$

where

$$\mathop{S}_{z_j}^{z_{j+1}} (f \oplus g)(\xi)\,d(q,\xi) = \begin{cases} (z_{j+1}-z_j)f(z_j)g(z_{j+1}); \text{for } z_{j+1}=(qx_j,y_j)\text{or}(x_j,qy \\ \\ (z_{j+1}-z_j)f(z_{j+1})g(z_j); \text{for } z_{j+1}=(q^{-1}x_j,y_j)\text{or}(x_j, \end{cases}$$

The next two theorems are q-analogues of monodiffric results given by Kurowski [81] and Berzsneyi [16,17]. The proc are similiar and so are omitted here.

THEOREM 3.7.1. If f is q-analytic and g is p-analytic in then

$$\mathop{S}_{C}(f \oplus g)(\xi)\ d(q,\xi) = 0$$

where C is any discrete closed curve in D.

THEOREM 3.7.2. If D is a finite discrete closed curve compri sing the set of boundary points of D. (This result is in fact the q-analogue of Green's Identity).

From this theorem, if f is q-analytic on a discrete domain D, then since $Lf = 0$, it follows that

$$\mathop{S}_{C}(f \oplus g)(\xi)d(q,\xi) = \Sigma_{D_q} f(\xi)B_g(q\xi) \qquad \text{......} (3.7.1)$$

A discrete function G_a is called a singularity functi if it satisfies

$$B[G_a(\xi)] = \begin{cases} 1; & \xi = a \\ 0; & \xi \neq a \end{cases} \quad \text{where} \quad a, \xi \in Q^1 \qquad \ldots\ldots (3.7.2)$$

If such a function exists then (3.7.1) would reduce to

$$\underset{C}{S} \, (f \oplus G_a)(\xi) d(q, \xi) = f(q^{-1} a)$$

an analogue of Cauchy's integral formula.

The following lemma gives the general form of G_a whi satisfies (3.7.2):

THEOREM 3.7.1. Let $a = (a_1, a_2)$ be a given point in Q^1. I $G_a(z) = G_a(q^m a_1, q^n a_2)$ is a discrete function in Q^1, given

$$G_a(q^m a_1, q^n a_2) = \begin{cases} \dfrac{\left[{m+n \atop n} \right]_q a_1^n a_2^m (a_2 + ia_1)}{(a_1 - ia_2)_{n+1} (a_2 + ia_1)_{m+1}} ; & m \geq 0, \ n \geq 0 \\ 0 ; & \text{all other integer values of } m, n, \end{cases}$$

then

$$B[G_a(z)] = \begin{cases} 1; & z = a = (a_1, a_2) \\ 0; & z \neq a, \quad z \in Q^1 \end{cases}$$

By combining the result of Lemma 3.7.1 with (3.7.1) th following analogue of Cauchy's Integral formula is obtained:

THEOREM 3.7.3. If f is q-analytic in D and if Ga is the singularity function defined above then

$$\underset{C}{S}(f \oplus G_{qa})(\xi)d(q,\xi) = \begin{cases} f(a); & a \in D_q \\ 0; & \text{otherwise} \end{cases}$$

where C is the discrete closed curve enclosing D.

8. DISCRETE ANALYTIC CONTINUATION BOUNDARY CONDITIONS. If a q-analytic function is defined on a subset of Q^1 then it has a unique extension, as a q-analytic function, to certain other points of the discrete plane. The q-difference operator L, defined in (3.4.1), involves the basic triad of points, $T(z) = \{z,(qx,y), (x,qy)\}$ and so from (3.4.2) it follows that if a q-analytic function is defined at any two points of $T(z)$ then it is uniquely determined at the third point. Some examp of continuation from a boundary which follow from this conditi are now given.

(i) If a function f is defined on the horizontal set of points $\{(q^m x,y);\ m \in Z\}$, then it can be uniquely continued as a q-analytic function to all points of Q^1 below this set (i.e. to all points of the form $\{(q^m x,q^n y);\ m \in Z;\ n = 0,1,2$

(ii) Similarly, if the function f is defined on the vertical set $\{(x, q^n y);\ n \in Z\}$, then f has a unique continuation as a q-analytic function to all points of Q^1 to the left of this set (i.e., to all points of the form

$\{(q^m x,\ q^n y);\ n \in Z;\ m = 0,1,2,\ldots..\})$.

(iii) If f is defined on the sets $\{(q^m x, y);\ m \in Z\}$ and $\{(x,\ q^n y);\ n = 0,-1,-2,\ldots.\}$, then it has a unique continuation as a q-analytic function to all points of Q^1.

(iv) If f is defined on $\{(q^m x, y);\ m = 0,1,2,\ldots..\}$ and $\{(x,\ q^n y); n = 0,1,2,\ldots..\}$, then f has a unique continuation into the rectangular region $\{(q^m x,\ q^n y);\ m = 0,1,2,\ldots\ldots;$ $n = 0,1,2,\ldots\ldots\}$.

(v) Let D be a finite dsicrete domain and f a q-analytic function defined on the set of boundary points of D (denoted by $\partial(D)$). The domain D consists of a union of basic sets i.e., a union of lattice points of the form $D = \{(q^i x, q^j y);\ i \in I, j \in J\}$ where I, J are sets of integers determined by D.

Define m_D, n_D and M_D by,

$$m_D = \min_{i \in I} i,\quad n_D = \min_{j \in J} j,\quad M_D = \max_{\substack{i \in I \\ j \in J}} (i+j)$$

and ℓ_1, ℓ_2, ℓ_3 by

$$\ell_1 = \{(q^{m_D} x, q^j y);\ j = n_D,\ n_D+1, \ldots\ldots M_D - m_D\}$$

$$\ell_2 = \{(q^i x,\ q^{n_D} y);\ i = m_D, m_D+1, \ldots\ldots M_D - m_D\}$$

$$\ell_3 = \{(q^i x,\ q^j y)\ ;\ i+j = M_D \text{ where } i = m_D,\ m_D+1, \ldots M_D - n_D$$

$$\text{and } j = n_D,\ n_D+1, \ldots\ldots, M_D - m_D\}.$$

Figure 2 illustrates the above notation. The boundary $\partial(D)$ is indicated by the solid line, ℓ_1 is given by the horizontal set of points between A and B, ℓ_2 by the vertical points between B and C and ℓ_3 is given by the diagonal like set of points between A and C. If G represents the subset of Q^1 bounded by and including ℓ_1, ℓ_2, ℓ_3 then the following theorem can readily be proved by repeated application of 2.4.2.

THEOREM 3.8.1. If a q-analytic function f is defined on the boundary $\partial(D)$ of some finite discrete domain D, then there exists a unique q-analytic function g, defined on G, such that $f = g$ on $\partial(D)$.

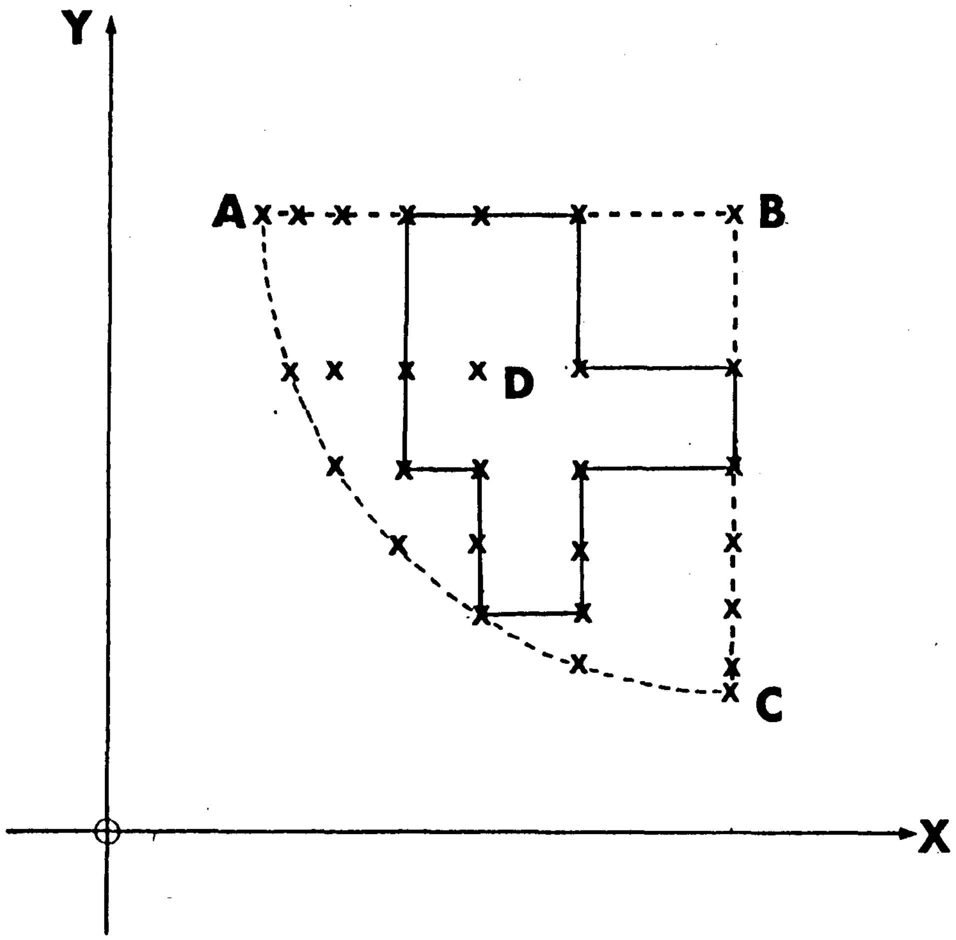

FIGURE 2

9. CONTINUATION FROM THE AXIS. The discrete plane Q^1 is now to be extended to include certain points on the axes. If (x^1, y^1) is the fixed points from which the lattice Q^1 is defined, then the sets X, Y are defined by $X = \{(q^m x', 0);\ m \in Z\}$, $Y = \{(0,\ q^n y');\ n \in Z\}$. The extended discrete plane $\overline{Q}$ is then defined as

$$\overline{Q} = Q^1 \cup X \cup Y \qquad \ldots\ldots (3.9.1)$$

The discrete rectangular domain R^1 is defined by

$$R^1 = \{(q^m x',\ q^n y');\ m = 0,1,2,\ldots\ldots;\ n = 0,1,2,\ldots\ldots\}$$

and if X^+, Y^+ are defined by $X^+ = \{(q^m x', 0);\ m = 0,1,2,\ldots\}$, $Y^+ = \{(0,\ q^n y');\ n = 0,1,2,\ldots\ldots\}$, then the extended rectangular domain R is defined as

$$\overline{R} = R^1 \cup X^+ \cup Y^+ \qquad \ldots\ldots (3.9.2)$$

Discrete functions can now be defined on X, Y. The values on the axes of a discrete function f defined on R^1, are given by

$$f(x,0) = \lim_{n \to \infty} f(x,\ q^n y'),\ f(0,y) = \lim_{m \to \infty} f(q^m x', y)$$

where $(x,y) \in R^1$. Alternatively this can be expressed as

$$f(x,0) = \lim_{y \to 0} f(x,y)$$
$$\qquad \ldots\ldots (3.9.3)$$
$$f(0,y) = \lim_{x \to 0} f(x,y)$$

where $(x,y) \in R'$.

The definition of q-analytic functions given in (3.3.2) is now extended to include functions defined on the axes X, Y. A function f is said to be q-analytic on X^+ if the limit in (3.9.3) exists for each x such that $(x,0) \in X^+$, and if

$$\lim_{y \to 0} \theta_y f(x,y) = \theta_x f(x,0); \qquad (x,y) \in R' \qquad \ldots\ldots (3.9.4)$$

Similarly, f is q-analytic on Y^+ if the limit in (3.9.3) exists and if

$$\lim_{x \to 0} \theta_x f(x,y) = \theta_y (0,y); \; (x,y) \in R' \qquad \ldots\ldots (3.9.5)$$

If f is q-analytic in R^1 and if (3.9.4), (3.9.5) hold then f is said to be q-analytic in $\bar{R}$. This definition can of course be extended to Q^1 and $\bar{Q}$, but for the present purposes the above suffices.

The q-difference operator of order j is defined by
$\theta^j[f(z)] = \theta[\theta^{j-1} f(z)]$; $\theta^o[f(z)] = f(z)$; $j = 0,1,2,\ldots\ldots.$

Analytic functions of a continuous complex variable have derivatives of all orders and a corresponding result is true for q-analytic functions defined on $\bar{R}$, as follows:

THEOREM 3.9.1. If f is q-analytic in $\bar{R}$, then,

(i) for $z \in R'$, $\theta_x^j f(z)$ and $\theta_y^j f(z)$ exist and are q-analytic for $j = 0,1,2\ldots\ldots$ (ii) for $(x,0) \in X^+$, $\lim_{y \to 0} \theta_y^j f(x,y)$ and $\theta_x^j f(x,0)$ exist and are q-analytic, and (ii) for $(0,y) \in Y^+$, $\lim_{x \to 0} \theta_x^j f(x,y)$ and $\lim_{y \to 0} \theta_y^j f(0,y)$ exist and are q-analytic.

A method is now derived whereby functions defined on the axes can be continued into the discrete plane as q-analytic functions.

If f is q-analytic in $\bar{R}$, then for $z \in R'$, it follows from (3.3.2) that

$$\theta_x f(z) = \theta_y f(z) = \frac{f(z) - f(x,qy)}{(1-q)iy} \qquad \ldots\ldots (3.9.6)$$

Hence $f(x,qy) = f(x,y) - (1-q)iy\, \theta_x f(x,y)$

$$= (1-(1-q)iy\, \theta_x)\, f(x,y),$$

and in general, using the notation of (1.3.8)

$$f(x,\ q^n y) = (1-(1-q)iy\,\Theta_x)_n\ f(x,y).$$

Since f is q-analytic in $\bar{R}$, it follows from (3.9.4) that

$\lim_{n\to\infty} f(x,q^n y) = f(x,0)$ exists and so,

$$f(x,0) = (1-(1-q)iy\,\Theta_x)_\infty\ f(x,y)$$

Assuming for the moment the formal symbolic methods can be used

$$f(x,y) = (1-(1-q)iy\,\Theta_x)_\infty^{-1}\ f(x,0) \qquad \text{......} \ (3.9.7)$$

Now a well known result from q-difference theory is the series expansion for the function $(1-a)_\infty^{-1}$ given by

$$\sum_{j=0}^{\infty} \frac{a^j}{(1-q)_j} \quad |a| < 1,$$

(See for example Hahn [47]). Applying this to (3.9.7) gives

$$f(x,y) = [\sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j}\ (iy)^j\,\Theta_x^j]\ f(x,0)$$

$$= \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!}\,\Theta_x^j\ f(x,0) \qquad \text{......} \ (3.9.8)$$

Where $[j]!$ is defined by (1.3.3) and $\Theta_x^j f(x,0)$ exists by theorem (3.9.1).

The methods used here in attempting to solve equation (3.9.6) have of course been formal symbolic ones. It remains to be verified that $f(x,y)$, as given by (3.9.8), in fact represents a solution of (3.9.6) when the series converges.

THEOREM 3.9.2. If $f(z)$ given by

$$f(x,y) = \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \Theta_x^j f(x,0)$$

is convergent for each $z = (x,y)$ in some rectangular domain R', then f is q-analytic in R', and hence satisfies Equation (3.9.6).

In fact more than the above is true. The series representation of f is also q-analytic on X^+.

THEOREM 3.9.3. If the series (3.9.8) is convergent in R^1, then

$$\lim_{y \to 0} f(x,y) = f(x,0) \quad \text{and} \quad \lim_{y \to 0} \Theta_y f(x,y) = \Theta_x f(x,0).$$

From the above then it is clear that functions defined on the x-axis can be continued (under conditions of convergence) into q-analytic functions defined on Q', by means of the operator C_y, where

$$\mathcal{C}_y = \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \theta_x^j \qquad \ldots\ldots (3.9.9)$$

The function f, given by

$$f(z) = \mathcal{C}_y[f(x,0)]$$

$$= \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \theta_x^j[f(x,0] \qquad \ldots\ldots (3.9.10)$$

is called the q-analytic continuation of $f(x,0)$ into a q-analytic function f defined at the point $z = (x,y) \in Q'$.

Similarly it can be shown that

$$\mathcal{C}_x[f(0,y)] = \sum_{j=0}^{\infty} \frac{x^j}{[j]!} \theta_y^j[f(0,y)] \qquad \ldots\ldots (3.9.11)$$

represents the q-analytic continuation from the y-axis.

In article 12, the operator $\mathcal{C}$ will be used to construct discrete analytic analogues of elementary functions. It can also be used to define an analogue for multiplication, but before doing so, some fundamental properties of are noted.

(i) If k is a scalar constant then $\mathcal{C}_y(k) = k$

(ii) $\mathcal{C}_y[kf(x,0)] = k\, \mathcal{C}_y[f(x,0)]$

(iii) If $\mathcal{C}_y[f(x,0)]$, $\mathcal{C}_y[g(x,0)]$ exists then
$\mathcal{C}_y[f(x,0)] + \mathcal{C}_y[g(x,0)] = \mathcal{C}_y[f(x,0) + g(x,0)]$.

(iv) If $f_n(x,0) \longrightarrow f(x,0)$ pointwise and the series representation of $\mathcal{C}_y[f_n(x,0)]$ is uniformly convergent in n, then

$$\lim_{n \to \infty} \mathcal{C}_y[f_n(x,0)] = \mathcal{C}_y[f(x,0)]$$

(v) It is interesting to note that $\mathcal{C}_y[f(x,0)]$ may be regarded as the q-analogue of the Taylor series expansion of an analytic function about a point on the x-axis. This can be verified by noting that

$$\lim_{q \to 1} [j]! = j! \quad \text{and} \quad \lim_{q \to 1} \Theta_x^j f(x,0) = f^{(j)}(x,0).$$

10. MULTIPLICATION OF q-ANALYTIC FUNCTIONS. If two q-analytic functions are multiplied in the usual way, the resultant function is not in general q-analytic. This it is desirable to devise an alternative operation, analogous to multiplication, which retains many of the usual algebraic properties of the classical multiplier of functions of a continuous variable.

In fact if two q-analytic functions f and g are multiplied, and the resultant fg is q-analytic, then it can readily be shown that either $f(z) = f(qx,y) = f(x,qy)$ or $g(z) = g(qx,y) = g(x,qy)$.

An analogue for multiplication of q-analytic functions can be defined as follows. Let R^1 denote a discrete rectangular domain and X^+ the corresponding points on the x-axis (see (3.9.1)). If the discrete functions f,g are q-analytic in R^1 and defined on X^+, then the convolution operator * is defined by

$$(f*g)(z) = \mathcal{C}_y[f(x,0)g(x,0)] = \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \theta_x^j[f(x,0)g(x,0)] \qquad \ldots\ldots (3.10.1)$$

for all $z \in R'$ such that the series converges.

In some respects, the method used above to obtain * is similar to the approach of Kurowski [81] who defined a multiplicative operator for monodiffric functions. However, his operator retained neither the commutative nor the associative law. It is now to be shown that * has most of the properties expected of a multiplicative operator and infact it forms, with addition, an integral domain over a wide class of q-analytic functions.

For convenience it is assumed that the functions under consideration are q-analytic in a discrete rectangular region $\bar{R}$ given by (3.9.2). If in addition the q-analytic function f has a convergent representation

$$f(z) = \mathcal{C}_y[f(x,0)] = \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \theta_x^j f(x,0)$$

for all $z \in \bar{R}$, then f is said to belong to the class A. If the series converges absolutely, f is said to belong to the class B.

The following fundamental properties of * can readily be established.

(i) If $f,g \in A$ then $(f*g)(z) = (g*f)(z)$

(ii) If $f,g,h \in A$ then

$$[f*(g+h)](z) = (f*g)(z) + (f*h)(z)$$

provided $f*g$ and $f*h$ exist.

(iii) If $f,g,h \in A$ then

$$[(f*g)*h](z) = (f*(g*h)](z)$$

provided $(f*g)*h$ exists.

(iv) If $I(z)$ is the identity function for all $z \in \bar{R}$ and if $f \in A$, then $(f*I)(z) = f(z)$, similarly if $k(z)$ is the constant function $k(z) = k$; $z \in \bar{R}$, then $(f*k)(z) = kf(z)$.

(v) If $f,g \in A$ such that $(f*g)(z) = 0$ for all $z \in \bar{R}$ then

$$\sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \theta_x^j [f(x,0)g(x,0)] = 0$$

and by the identity theorem for the power series in y, the case $j = 0$ leads to $f(x,0)\, g(x,0) = 0$. Hence either $f(x,0) = 0$ or $g(x,0) = 0$ and for $f(x,0)$ it follows that

$$f(z) = \mathcal{C}_y[f(x,0) = \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \theta_x^j [0] = 0.$$

So for q-analytic functions of class A, the operator * has no non-zero divisors of zero.

In the above properties it was assumed that q-analytic functions f and g belong to the class A, which implies the existence of $\mathcal{C}_y f(x,0)$ and $\mathcal{C}_y g(x,0)$ in $\bar{R}$. However this condition does not necessarily guarantee the existence of $(f*g)(z)$. The operator * is infact closed for functions of class B for which the following lemma is required:

LEMMA 3.10.1. $\theta_x^j = q^{kj}\, \theta_{x_1}^j$, where $x_1 = q^k x$.

THEOREM 3.10.1. If $f, g \in B$ then $(f*g) \in B$.

It has been shown then that with respect to the convolution operator * an ordinary addition +, the class B of q-analytic functions is closed, commutative, distributive, associative, has an identity and has no non-zero divisor of zero. Hence the following has been demonstrated.

THEOREM 3.10.2. The algebraic system $\{B, +, *\}$ is an integral domain.

THEOREM 3.10.3. If $f, g \in B$, then

$$\theta^n[(f*g)(z)] = \sum_{k=0}^{n} \left[{n \atop k}\right]_q q^{k(k-n)} [\theta^k f(z)]*[\theta^{n-k} g(q^k z)].$$

Of immediate consequence is a formula for integration by parts.

COROLLARY 3.10.1. If $f, g \in B$; $a, b, z \in \bar{R}$, then

$$\underset{a}{\overset{b}{S}} [f(z)* \theta_g(z)]d(q,z) = [f(z)*g(z)]_a^b - \underset{a}{\overset{b}{S}}[g(qz)*\theta f(z)]d(q,z).$$

From the preceding properties of the convolution * it has been seen that a close analogy with ordinary multiplication is evident.

11. DISCRETE POWERS. The development of a study of power series representations for discrete analytic functions has been hampered by the difficulty in obtaining a convenient analogue of the function z^n. A formula has been found by Meredov [89], but it is rather complicated for explicit use. Isaacs [69] defined a monodiffric analogue of z^n and found certain results for polynomial and power series. A discrete analogue $z^{(n)}$ of the classical function z^n is now obtained for the q-analytic

function theory and is shown to have concise properties which lead to important applications.

In article 10, the operator $\mathcal{C}_y$ has been used to define the multiplicative operator * by means of a continuation from the real axis into the discrete plane Q'. A similar is now being used to define $z^{(n)}$ for non-negative integers n.

The operator θx has the property $\theta_x(x^n) = [n]x^{n-1}$ (see (1.3.1) (3.3.1)) and a q-analytic function $Z^{(n)}$ will denote the discrete analogue of z^n if it satisfies the condition: (i) $\theta_x(z^{(n)}) = [n]z^{(n-1)}$, (ii) $z^{(0)} = 1$ (iii) $0^{(n)} = 0$, $n \geq 1$.

In fact $z^{(n)}$ (n a non-negative integer) is to be defined by :

$$z^{(n)} = \mathcal{C}_y(x^n) = \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \theta_x^j(x^n)$$

and since

$$\theta_x^j(x^n) = \begin{cases} (1-q^{n-j+1})(1-q^{n-j+2})\ldots\ldots(1-q^n)x^{n-j} & ;\ j \leq n \\ 0: & j > n \end{cases}$$

it follows that

$$z^{(n)} = \sum_{j=0}^{n} [{}^{n}_{j}]_q (iy)^j x^{n-j}$$

...... (3.11.1)

$$= \sum_{j=0}^{n} [{}^{n}_{j}]_q x^j (iy)^{n-j} \text{ (Since } [{}^{n}_{j}]_q = [{}^{n}_{n-j}]_q)$$

Since θ is a linear operator it follows that

$$\theta_x z^{(n)} = \sum_{j=0}^{n} [{}^{n}_{j}]_q \theta_x(x^{n-j})(iy)^j$$

$$= \sum_{j=0}^{n-1} [{}^{n}_{j}]_q [n-j] x^{n-j-1}(iy)^j$$

$$= \frac{(1-q^n)}{(1-q)} \sum_{j=0}^{\infty} [{}^{n-1}_{\ j}]_q x^{n-1-j}(iy)^j$$

$$= [n] z^{(n-1)}$$

Similarly $\theta_y z^{(n)} = [n] z^{(n-1)}$ and so $z^{(n)}$ is q-analytic ($\theta_x z^{(n)} = \theta_y z^{(n)}$) and satisfies condition (i) above. condition (ii) and (iii) are trivially satisfied and hence the function $z^{(n)}$ is an appropriate discrete analogue of z^n.

Since

$$\lim_{q \to 1} [{}^{n}_{j}]_q = ({}^{n}_{j})$$

and from the fact that

$$\left[{n \atop j}\right]_q = \left[{n \atop n-j}\right]_q$$

it follows that the expression for $z^{(n)}$ given in (3.11.1) is remarkably similar to the binomial expansion for $z^n = (x+iy)^n$. Note also, $\lim\limits_{q \to 1} z^{(n)} = z^n$. In many respect the function $z^{(n)}$ bears a closer resemblance to z^n than the monodiffric analogue considered by Isaacs [69].

It is also interesting to note the similarity between $z^{(n)}$ and the function $(x+iy)_n$ which is given by:

$$\begin{aligned}(x+iy)_n &= (x+iy)(x+iqy)\ldots\ldots(x+iq^{n-1}y) \\ &= \sum_{j=0}^{n} \left[{n \atop j}\right]_q q^{j(j-1)/2} \; x^{n-j}(iy)^j\end{aligned}$$

An important property of the function $z^{(n)}$ is that the additive law for indices holds with respect to the convolution operator $*$. In fact if m,n are non-negative integers, then by (3.10.1)

$$z^{(m)} * z^{(n)} = \mathcal{C}_y[(x,0)^{(m)} \; (x,0)^{(n)}]$$

By the definition of $z^{(n)}$, it follows that $(x,0)^{(m)} = x^m$, and hence

$$z^{(m)} * z^{(n)} = \mathcal{C}_y[x^m x^n] = \mathcal{C}_y[x^{m+n}] = z^{(m+n)}.$$

The following additional properties are noted without proof:

(a) $\lim\limits_{z \to 0,\, z \in Q'} z^{(n)} = 0$

(b) $z^{(n)} = \lim\limits_{z_0 \to 0,\, z_0 \in Q'} \mathop{S}\limits_{z_0}^{z} \xi^{(n-1)} d(q, \xi) = \mathop{S}\limits_{0}^{z} \xi^{(n-1)} d(q, \xi)$;

(c) $(\lambda z)^{(n)} = \lambda^n z^{(n)}$;

(d) $\lim\limits_{n \to \infty} z^{(n)} = 0$ if and only if $\| z \| < 1$,

and

$\lim\limits_{n \to \infty} |z^{(n)}| = \infty$ if and only if $\| z \| > 1$;

where

$$\| z \| = \max \{|x|, |y|\};\ z \in Q^1.$$

A discrete analogue of the function $(z-z_0)^n$ is of course necessary if power series expansions about arbitrary points in the plane are to be considered. The q-analogue $(z-z_0)^{(n)}$, $z_0 \in Q^1$, can be defined by:

$$(z-z_0)^{(n)} = \sum_{j=0}^{n} \left[{n \atop j}\right]_q (x-x_0)_{n-j}\, i^j (y-y_0)_j; \qquad \ldots\ldots (3.11.2)$$

and it can be readily verified that this function is q-analytic and satisfies:

(i) $\theta(z-z_o)^{(n)} = [n](z-z_o)^{(n-1)}$; (ii) $(z-z_o)^{(0)} = 1$; and (iii) $0^{(n)} = 0$, $n \geq 1$.

It is clear that $(z-z_o)^{(n)}$ is consistent with the definition of $z^{(n)}$ to which it reduces when $z_o = 0$. Also;

$$\lim_{q \to 1} (z-z_o)^{(n)} = (z-z_o)^n.$$

It may be noted that unlike the case of $z^{(n)}$, the operator * does not in general preserve the property of addition of indices for discrete powers $(z-z_o)^{(n)}$. However, if z_o is purely real or imaginary, it follows that:

$$(z-z_o)^{(m)} * (z-q^m z_o)^{(n)} = (z-z_o)^{m+n}.$$

For example, if $z_o = x_o$; x_o real, then

$$(z-x_o)^{(n)} = \sum_{j=0}^{n} \left[{n \atop j}\right]_q (x-x_o)_{n-j}(iy)^j$$

$$= \sum_{j=0}^{n} \frac{(1-q)^j}{(1-q)_j} (iy)^j \theta_x^j (x-x_o)_n$$

$$= C_y[(x-x_o)_n]$$

Hence

$$(z-x_o)^{(m)}*(z-q^m x_o)^{(n)} = \mathcal{C}_y[(x-x_o)_m(x-q^m x_o)_n]$$

$$= \mathcal{C}_y[(x-x_o)_{m+n}] = (z-x_o)^{(m+n)}.$$

Analogue for the case of arbitrary exponents involve infinite series and are treated in Section 13.

12. DISCRETE EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS. In the monodiffric theory, Isaacs [69] obtained as an analogue of the exponential function the function $2^x\ (1+i)^y$ which satisfies the monodiffric equation $\Delta f(z) = f(z)$. Duffin [32] derived a similar function $3^x \{(2+i)/(a-i)\}^y$ which satisfies $\Delta f(z) = f(z$ for discrete analytic functions of the second kind. Apart from a few simple properties, a very limited study has been made of these functions.

In this section a q-analytic function $e(z)$ is defined (the analogue of e^z); a reciprocal function $E(z)$ is derived such that $e(z)*E(z) = 1$, and using these two functions results are obtained for q-analogues of trigonometric functions and identities.

The function $e_q(x)$ defined by:

$$e_q(x) = \frac{1}{(1-x)_\infty}; \qquad x \neq q^{-n},$$

n, a non-negative integer,

$$= \sum_{j=0}^{\infty} \frac{x^j}{(1-q)_j}; \ |x| < 1,$$

has been studied by Hahn [47] and has the properties

$\Theta_x \, e_q(x) = e_q(x)$ and $\lim_{q \to 1} e_q \{(1-q)x\} = e^x$.

The function $e(z)$ is to be defined by:

$$e(z) = \mathcal{C}_y \, e_q \{(1-q)x\} .$$

Hence

$$e(z) = \mathcal{C}_y[\sum_{k=0}^{\infty} \frac{(1-q)^k}{(1-q)_k} x^k]; \qquad |x| < 1/(1-q),$$

$$= \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \Theta_x^j [\sum_{k=0}^{\infty} \frac{(1-q)^k}{(1-q)_k} x^k]$$

and so by corollary 3.4.2 it follows that

$$e(z) = \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \sum_{k=0}^{\infty} \frac{(1-q)^{k-j}}{(1-q)_k} (1-q^{k-j+1})_j \, x^{k-j}$$

$$= \sum_{j=0}^{\infty} \frac{(iy)^j}{[j]!} \sum_{k=0}^{\infty} \frac{x^k}{[k]!},$$

the two series being absolutely convergent if

$|x| < 1/(1-q)$ and $|y| < 1/(1-q)$. Hence;

$$e(z) = e_q\{(1-q)iy\}\ e_q\{(1-q)x\}.$$

For values of x,y outside the domain of convergence, $e(z)$ can be expressed using the analytic continuation form for e_q:

$$e(z) = \frac{1}{(1-(1-q)x)_\infty\,(1-(1-q)iy)_\infty}$$

The following theorem justifies the definition of $e(z)$.

THEOREM 3.12.1. The function $e(z)$ is q-analytic; it satisfies $\Theta\, e(z) = e(z)$ and infact for $\| z \| < 1/(1-q)$

$$e(z) = \sum_{k=0}^{\infty} \frac{z^j}{[j]!}.$$

Convergence of the above series for $e(z)$ is restricted to the region $\| z \| < 1/(1-q)$ whereas e^z is entire. However for q close to unity the restriction becomes less important. Note also that $\lim_{q \to 1} e(z) = e$.

The function $Eq(x)$ is defined by $1/eq(x)$ (see Hahn [47]) and if the q-analytic function E is defined by

$$E(z) = \mathcal{C}_y[Eq\{(1-q)x\}]$$

if follows that

$$(e*E)(z) = \mathcal{C}_y[e(x,0)E(x,0)]$$

$$= \mathcal{C}_y[eq\{(1-q)x\} Eq\{(1-q)x\}]$$

$$= \mathcal{C}_y(1) = 1$$

Hence with respect to the operator *, E and e are reciprocal functions.

The series expansion for $Eq(x)$ is given by

$$Eq(x) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} \frac{x^k}{[k]!} \quad \text{(See Hahn [47])}$$

where the series converges for all x. By means of rearrangement of series techniques-similar to ones used in finding the series representation of $e(z)$, it can be readily shown that

$$E(z) = \sum_{j=0}^{\infty} (-1)^j q^{j(j-1)/2} \frac{z^{(j)}}{[j]!},$$

the series being convergent for all z.

The analogues of sine and cosine can be defined as:

$$s(z) = \frac{1}{2i}\,[e(iz) - e(-iz)]$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{z^{(2j+1)}}{[2j+1]!}, \quad \|z\| < 1/(1-q),$$

and

$$C(z) = \frac{1}{2}\,[e(iz) + e(-iz)]$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{z^{(2j)}}{[2j]!} \quad \| z \| < 1/(1-q)$$

and it can be verified that $s(z)$ and $c(z)$ are q-analytic and satisfy the q-difference equation

$$\theta^2 f(z) = -f(z)$$

Alternatively, discrete analogues of sine and cosine can be defined by,

$$S(z) = \frac{1}{2i}\,(E(-iz) - E(iz)]$$

$$= \sum_{j=0}^{\infty} (-1)^j q^{j(2j+1)} \frac{z^{(2j+1)}}{[2j+1]!}$$

and

$$C(z) = \frac{1}{2}\,[E(iz) + E(-iz)]$$

$$= \sum_{j=0}^{\infty} (-1)^j q^{j(2j-1)} \frac{z^{(2j)}}{[2j]!} .$$

It follows that $S(z)$ and $C(z)$ are q-analytic and are solutions of the q-difference equation

$$\theta^2 f(z) = -qf(q^2 z)$$

The following trigonometric identities can readily be verified:

$$(C \times c)(z) + (S \times s)(z) = 1$$

$$(C * s)(z) - (S * c)(z) = 0.$$

Also from the above definitions it is clear that in the limit as $q \longrightarrow 1$; $e(z) \longrightarrow e^z$; $E(z) \longrightarrow e^{-z}$; $s(z)$ and $S(z) \longrightarrow \text{Sin } z$; $C(z)$ and $C(z) \longrightarrow \text{Cos } z$.

13. THE DISCRETE FUNCTIONS $(z-z_0)^{(a)}$. Discrete analogues of the function $z^{(a)}$ where a is an arbitrary constant have been difficult to find. In [53] and [54] the author examined the monodiffric function $z^{(a)}$ and outlined certain results. In q-difference theory the function $(x-x_0)_a$ is defined by

$$(x-x_0)_a = x^a \frac{(1-x_0/x)_\infty}{(1-q^a x_0/x)_\infty}$$

and satisfies $\theta_x(x-x_0)_a = [a]\,(x-x_0)_{a-1}$.

The discrete function $(z-z_o)^{(a)}$, $z \in Q^1$ is to be defined hereby:

$$(z-z_o)^{(a)} = \sum_{j=0}^{\infty} \binom{a}{j}_q i^j (y-y_o)_j (x-x_o)_{(a-j)}$$

where a is an arbitrary scalar constant. It can readily be shown that this series converges absolutely for all $z \in Q^1$ such that $y < |x_o|$. Moreover the function f is q-analytic in the region of convergence and satisfies the requirements: (i) $\Theta(z-z_o)^{(a)} = [a](z-z_o)^{(a-1)}$; (ii) $(z-z_o)^{(0)} = 1$; (iii) $0^{(a)} = 0$, $a > 0$.

Hence the function is a q-analytic analogue of $(z-z_o)^a$ and furthermore in the special case where a is a non-negative integer, the series reduces to the one given in [3.11.2]. Also, since

$$\lim_{q \to 1} \binom{a}{j}_q = \binom{a}{j}$$

and $\lim_{q \to 1} (x-x_o)_{a-j} = (x-x_o)^{a-j}$

it follows that

$$\lim_{q \to 1} (z-z_o)^{(a)} = (z-z_o)^a.$$

14. SERIES REPRESENTATION OF q-ANALYTIC FUNCTIONS. An analogue of Taylor's series is now derived for q-analytic functions.

THEOREM 3.14.1. If f is q-analytic in a discrete rectangular domain R^1, then for any $z_0 \in R^1$,

$$f(z) = \sum_{j=0}^{\infty} \frac{\theta^j[f(z_0)]}{[j]!} (z-z_0)^{(j)}$$

for all z in the set A defined by

$$A = \{(q^m x_0, q^n y_0); m \geq 0, n \geq 0\} .$$

A more general theorem is now stated.

THEOREM 3.14.2. If f is q-analytic in some rectangular domain R^1 which includes the set A, then

$$f(z) = \sum_{j=0}^{\infty} \frac{\theta^j[f(z_0)]}{[j]!} (z-z_0)^{(j)}$$

for any $z \in R^1$ for which the series converges absolutely.

The case $z_0 = 0$ raises special problems. To include the point $z = (0,0)$ extend the definition of $\bar{R}$ (from (2.9.2)) to $\bar{R}_0$ as follows:

$\bar{R}_o = \bar{R}\, U(0,0)$.

A discrete function f is said to be q-analytic on $\bar{R}_o$ if it is q-analytic on $\bar{R}$ and if in addition

$$\theta^j f(0) = \lim_{(x,y)\longrightarrow(0,0)} \theta^j[f(z)] \qquad \ldots\ldots \ (3.14.1)$$

exists.

Under certain conditions a discrete Maclaurin's series can be shown to represent a q-analytic function, provided convergence conditions are met. For example the following result can be demonstrated.

THEOREM 3.14.3. Let f be q-analytic in $\bar{R}_o$. If in addition $f(z) = \mathcal{C}_y[f(x,0)] = \mathcal{C}_y[f(x,0)] = \mathcal{C}_x[f(0,y)]$, the series representations of $\mathcal{C}_y$, $\mathcal{C}_x$ being uniformly and absolutely convergent in $\bar{R}_o$, then it follows that

$$f(z) = \sum_{j=0}^{\infty} \frac{\theta^j f(0)}{[j]!} z^{(j)}$$

where the series converges absolutely for all $z \in \bar{R}_o$.

IV CHAPTER

DISCRETE q-HYPERGEOMETRIC FUNCTIONS

1. INTRODUCTION. The function z^n is of basic importance in complex analysis since its use in infinite series leads to the Weierstrassian concept of an analytic function. It is desirable then to obtain a discrete analytic analogue of z^n and use it to expand discrete functions in power series.

The earlier notable attempts in this direction are due to Duffin [32], Duffin and Peterson [38] and Isaacs [69].

In 1972, Harman [52] in his doctoral thesis defined, for a non-negative integer n, a q-analytic function $z^{(n)}$ to denote the discrete analogue of z^n, if it satisfies the following conditions:

$$\left.\begin{aligned} D_q[z^{(n)}] &= \frac{(1-q^n)}{(1-q)}\, z^{(n-1)} \\ z^{(0)} &= 1 \\ 0^{(n)} &= 0,\ n > 1 \end{aligned}\right\} \qquad \ldots\ldots (4.1.1)$$

The operator C_y, denoted by

$$C_y = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D^j_{q,x} , \qquad \ldots\ldots (4.1.2)$$

when applied to the real function x^n, yields $z^{(n)}$. In fact, Harman defined $z^{(n)}$ by

$$z^{(n)} \equiv C_y (x^n); \; n \text{ a non-negative integer}$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D^j_{q,x} (x^n), \qquad \ldots\ldots (4.1.3)$$

which on simplification, yields

$$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j}_q x^{n-j} (iy)^j \qquad \ldots\ldots (4.1.4)$$

or alternatively

$$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j}_q x^j (iy)^{n-j} \qquad \ldots\ldots (4.1.5)$$

To justify that $z^{(n)}$ is a proper analogue of z^n, Harman [52] proved that $z^{(n)}$ is a q-analytic function and satisfies the three requirements of (4.1.1).

He also pointed out the similarity between the $z^{(n)}$ function and $(x + iy)_n$ defined by

$$(x + iy)_n = (x + iy)(x + iqy)(x + iq^2 y)\ldots\ldots(x + iq^{n-1} y)$$

$$= \sum_{j=0}^{n} \binom{n}{j}_q q^{\frac{1}{2}j(j-1)} x^{n-j} (iy)^j .$$

Using Harman's discrete analogue $z^{(n)}$ for the classical function z^{n}, Khan [76], in 1977, defined a discrete analogue of the q-hypergeometric functions ${}_r\phi_s^{(q)}[(a_r);(b_s);\ z]$ and besides discussing its elementary properties he obtained various elegant transformations and expansion formulae for these discrete hypergeometric function.

2. DISCRETE q-HYPERGEOMETRIC FUNCTIONS. It is well known that

$$D_q\left\{{}_r\phi_s^{(q)}\ [(a_r);\ (b_s);\ x]\right\}$$

$$= \frac{(1-q^{a_1})(1-q^{a_2})\ldots.(1-q^{a_r})}{(1-q)(1-q^{b_1})(1-q^{b_2})\ldots.(1-q^{b_s})}\ {}_r\phi_s^{(q)}[1+(a_r);1+(b_s);\ x],$$

and so it seems reasonable to assume that for n, a non-negative integer, a q-analytic function ${}_rM_s[(a_r);(b_s);q,z]$ will denote the discrete analogue of ${}_r\phi_s^{(q)}[(a_r);(b_s);x]$ if it satisfies the following conditions:

(i) $D_q\ {}_rM_s[(a_r);(b_s);q,z]$

$$= \frac{(1-q^{a_1})(1-q^{a_2})\ldots\ldots(1-q^{a_r})}{(1-q)(1-q^{b_1})(1-q^{b_2})\ldots.(1-q^{b_s})}\ {}_rM_s[1+(a_r);1+(b_s);q,z]$$

(ii) The first term of the series is 1.

(iii) ${}_rM_s[(a_r);(b_s);q,0] = 1$ ...... (4.2.1)

Such a function is obtained by applying the operator C_y defined in (4.1.2) to the q-Hypergeometric function ${}_r\phi_s^{(q)}[(a_r);(b_s);x]$, with real argument x.

In fact, ${}_rM_s[(a_r);(b_s);q,z]$ is defined by

$$ {}_rM_s[(a_r);(b_s);q,z] \equiv C_y\left\{ {}_r\phi_s^{(q)}[(a_r);(b_s);x] \right\} $$

$$ = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n \, z^{(n)}}{(q)_n \, (q^{(b_s)})_n} \qquad \text{...... (4.2.2)} $$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{(a_r)})_{n+k} \, x^n (iy)^k}{(q)_n (q)_k \, (q^{(b_s)})_{n+k}} \qquad \text{...... (4.2.3)} $$

The following theorem shows that ${}_rM_s[(a_r);(b_s);q,z]$ satisfies (4.2.1) and hence can be taken as a discrete analogue of ${}_r\phi_s^{(q)}[(a_r);(b_s);z]$;

THEOREM 4.2.1. ${}_rM_s[(a_r);(b_s);q,z]$ is q-analytic and satisfies the three requirements of (4.2.1).

It is of interest to note the similarity of ${}_rM_s[(a_r);(b_s);q,z]$ to the function ${}_r\phi_s^{(q)}[(a_r);(b_s);[x+iy]]$ defined by Jackson [74(a)] as follows:

$$ {}_r\phi_s^{(q)}[(a_r);(b_s);[x+iy]] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{(a_r)})_{m+n}\, x^m (iy)^n q^{\frac{1}{2}n(n-1)}}{(q)_m (q)_n (q^{(b_s)})_{m+n}} $$

$$ = \sum_{N=0}^{\infty} \frac{(q^{(a_r)})_N}{(q)_N (q^{(b_s)})_N} (x+iy)(x+iqy)\ldots\ldots(x+iq^{n-1}y) \qquad \ldots\ldots (4.2.4) $$

The discrete q-hypergeometric function defined in (4.2.2) can be written in either of the following two forms:

$$ {}_rM_s[(a_r);(b_s);q,z] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n (iy)^n}{(q)_n (q^{(b_s)})_n} {}_r\phi_s^{(q)}[(a_r)+n;(b_s)+n;x] \qquad \ldots\ldots (4.2.5) $$

or alternatively as,

$$ {}_rM_s[(a_r);(b_s);q,z] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n x^n}{(q)_n (q^{(b_s)})_n} {}_r\phi_s^{(q)}[(a_r)+n;(b_s)+n;iy] \qquad \ldots\ldots (4.2.6) $$

Note from (4.2.5), that ${}_rM_s[(a_r);(b_s);q,z]$, for $x = 0$, reduces to ${}_r\phi_s^{(q)}[(a_r);(b_s);iy]$ while for $y = 0$, becomes ${}_r\phi_s^{(q)}[(a_r);(b_s);x$

3. ELEMENTARY PROPERTIES. The following particular cases of (4.2.5-6) are noteworthy:

$$ {}_0M_0\,[—;\ —;\ q,z] = e_q(x)\ e_q(iy) \qquad \ldots\ldots\ (4.3.1)$$

$$ {}_0M_1\,[—;a;q,z] = (q)_{a-1}\left(\frac{-i}{\sqrt{x}}\right)^{a-1} \sum_{n=0}^{\infty} \frac{(y/\sqrt{x})^n}{(q)_n}\ q^{J_{a+n-1}}(2i\sqrt{x}), \qquad \ldots\ldots\ (4.3.2)$$

$$ {}_2M_1\,[a,b;c;q,z]$$

$$ = \frac{1}{(1-x)_{a+b-c}} \sum_{n=0}^{\infty} \frac{(q^a)_n\,(q^b)_n\,(iy)^n}{(q)_n(q^c)_n(xq^{a+b-c})_n}\ {}_2\phi_1\left[\begin{matrix} q^{c-a},q^{c-b};xq^{a+b-c+n} \\ q^{c+n} \end{matrix}\right] \qquad \ldots\ldots\ (4.3.3)$$

From (4.2.5-6) one observes that a discrete hypergeometric function can be regarded as a 'generating function' for the q-hypergeometric function of the form

$$ {}_r\phi_s^{(q)}\,[(a_r)+n;(b_s)+n;x] \text{ or } {}_r\phi_s^{(q)}\,[(a_r)+n;(b_s)+n;iy].$$

Further, summing up the above ${}_r\phi_s$-functions by means of known summation theorems, Khan [76] obtained

$$ {}_1M_0[a;—;q,z] = \frac{1}{(1-x)_a}\ {}_1\phi_1[q^a;xq^a;\frac{iy}{[1-xq^a]}], \qquad \ldots\ldots\ (4.3.4)$$

$$= \frac{1}{(1-iy)_a}\ {}_1\phi_1[q^a;\ iyq^a;\ \frac{x}{[1-iyq^a]}] \qquad \ldots\ldots (4.3.5)$$

${}_2M_1[a,-n;b;q,(q,y)]$

$$= \frac{(q^{b-a})_n\, q^{an}}{(q^b)_n}\ {}_2\phi_1\left[\begin{matrix} q^a, q^{-n}\ ;\ -iyq^{1-b} \\ q^{1+a-b-n}\ ;\ q^{-1} \end{matrix}\right] \qquad \ldots\ldots (4.3.6)$$

He also noted the following simple integral representations for ${}_2M_1[a,b;c;q,z]$ and ${}_3M_2[a,b,c;d,e;q,z]$:

$$\int_0^1 t^{b-1}(1-qt)_{c-b-1}\ {}_1M_0[a;-;q,zt]\,d(t;q)$$

$$= \sum_{n=0}^{\infty} \frac{(q^a)_n\,(z^{(n)}}{(q)_n} \int_0^1 t^{n+b-1}\ (1-qt)_{c-b-1}\ d(t;q)$$

$$= \frac{\Gamma_q(b)\ \Gamma_q(c-b)}{\Gamma_q(c)}\ {}_2M_1[a,b;c;q,z] \qquad \ldots\ldots (4.3.7)$$

provided $R(b) > 0$, $|x| < 1$, $|y| < 1$,

$$\int_0^1 \int_0^1 t^{b-1}\ (1-qt)_{d-b-1}\ \vartheta^{c-1}\ (1-q\vartheta)_{e-c-1}$$

$${}_1M_0[a;—;\ xt\vartheta]\,d(t;q)\,d(\vartheta;q)$$

$$= \frac{\Gamma_q(b)\Gamma_q(c)\Gamma_q(d-b)\Gamma_q(e-c)}{\Gamma_q(d)\ \Gamma_q(e)} {}_3M_2[a,b,c;d,e;q,z] \ \ldots\ldots \ (4.3.8),$$

provided $Rl(b) > 0$, $Rl(c) > 0$, $|x| < 1$, $|y| < 1$. Similarly, integral representation for ${}_rM_s$-function can be obtained.

V CHAPTER

DISCRETE ANALYSIS ON A RADIAL LATTICE

1. INTRODUCTION. The lattice of definition for discrete models of analytic functions is usually considered to be the set of Gaussian integers, and discrete analogue of the function z^n (called pseudo-powers) are of fundamental importance to the study of such theories.

In 1977 Zeilberger [129] extended the work of Duffin [32] and Duffin and Peterson [38] by defining a new system of pseudo-powers and thereby enlarging the class of discrete analytic functions which could be represented as an infinite series of pseudo-powers. However Zeilberger proved that no basis for discrete analytic function could be found such that the convergence of $\sum_{n=0}^{\infty} a_n z^n$ on C (Complex plane) ensures the convergence of $\sum_{n=0}^{\infty} a_n \pi_n(z)$ on the whole lattice (where π_n denotes the pseudo-power).

Pseudo-powers for monodiffric functions were defined by Isaacs [69] and the problem of representing a monodiffric function by a series of basis elements was considered by Atadzanov [11(a)].

However, the question-whether or not the pseudo-powers form a basis for monodiffric functions - was not answered.

Various alternative lattices of definition have been considered. For example, Duffin [33], considered a lattice of rhombs; Kurowski [79], a semidiscrete lattice; and Harman [56], geometric-difference spacing between points of a rectangular lattice. In 1981 Harman constructed a discrete model for analytic functions using lattice points of the complex plane arranged in radial form. In this chapter Harman's [58] radial lattice is considered with the origin as a cluster point. By making first order approximations to the polar form of the Cauchy Riemann conditions, the discrete analytic functions are defined as solutions of a finite difference approximations to the polar Cauchy-Riemann equations. In the resulting theory the discrete pseudo power $z^{(n)}$ (an analogue of z^n) has a simple algebraic form (a direct analogue of $\rho^n \exp(in\theta)$) and has ome interesting properties. Not only does the convergence of $\sum_{n=0}^{\infty} a_n z^n$ on C guarantee the convergence of $\sum_{n=0}^{\infty} a_n z^{(n)}$ on the lattice, but also the set $\{z^{(n)}\}_{-\infty}^{\infty}$ forms a basis (every function, discrete analytic on the lattice, can be written as a convergent series $\sum_{n=-\infty}^{\infty} a_n z^{(n)}$).

The difficulty in finding suitable analogues of z^n, where n is a negative integer has prevented the development of discrete Laurent series representations for functions with singularities. The analogue $z^{(n)}$ due to Harman is valid for all values for n and as the basis result indicates, a function with a singularity at the origin can be modelled by discrete analytic functions on the radial lattice.

Further significance of this theory is indicated by an analogue of the fundamental theorem of algebra. A convenient analogue of multiplication is defined which leads to a factorization of a discrete polynomial $\sum_{n=0}^{m} a_n z^{(n)}$ in terms of the zeros of the polynomial $\sum_{n=0}^{m} a_n z^{(n)}$.

2. DEFINITIONS AND NOTATIONS. The lattice of definition for the discrete functions to be considered here will consist of a certain set P of points given in polar co-ordinates.

Firstly, constants q and δ are chosen such that $0 < q < 1$ and $\delta = 2\pi/M$, where M is some fixed positive integer. The lattice P is then defined by

$P = (\rho,\theta); \rho \in q^n, n \in Z$ (the set of integers); $\theta \in m\delta$, $m = 0,1,2,\ldots\ldots\}$; where (ρ,θ) are the polar coordinates of the point $z = \rho \exp\{ i\theta \}$. Hence the points of the lattice are spaced in a geometric sequence radially (with the origin as cluster point) while the angle increment δ is fixed.

The geometric difference (or q-difference) operator and the forward difference operator Δ_θ are defined by

$$\Delta_\rho f(\rho,\theta) = [f(\rho,\theta) - f(q\rho,\theta)]/(1-q)\rho ,$$

$$\Delta_\theta f(\rho,\theta) = [f(\rho,\theta+\delta) - f(\rho,\theta)/\delta ; \qquad \ldots\ldots (5.2.1)$$

where $(\rho,\theta) \in P$.

A function f defined on the polar lattice P is said to be a discrete function and such a function is said to be discrete analytic (in the polar sense) at the point (ρ,θ) if

$$\begin{aligned} \rho \Delta_\rho u(\rho,\theta) &= \Delta_\theta v(\rho,\theta) \\ \Delta_\theta u(\rho,\theta) &= -\rho \Delta_\rho v(\rho,\theta) \end{aligned} \qquad \ldots\ldots (5.2.2)$$

where u, v are respectively the real and imaginary parts of f. In this way, the partial derivatives in the polar form of the Cauchy-Riemann equations have been replaced by partial difference operators. From the above two conditions, it can readily be shown that the equation

$$(\rho\Delta_\rho + i\Delta_\theta) f(\rho,\theta) = 0 \qquad \ldots\ldots (5.2.3)$$

provides a condition equivalent to (5.2.3) for a function to be discrete analytic at the point (ρ,θ).

Using the above definitions and methods similar to those used by Harman in his earlier paper [56] he defined a discrete integral which satisfies analogues of Cauchy theorems in portions of the lattice. However, the approach taken here is towards a characterisation of discrete analytic functions by means of power series.

3. DISCRETE ANALYTIC CONTINUATION AND MULTIPLICATION. If a discrete function is discrete analytic at all points of the lattice P, then from (5.2.1), (5.2.3) it follows that

$$\begin{aligned} f(\rho,\theta+\delta) &= f(\rho,\theta) + i\delta\rho\Delta_\rho f(\rho,\theta) \\ &= [1+i\delta(\rho\Delta_\rho)] \quad f(\rho,\theta), \end{aligned}$$

and in general $f(\rho,\theta + m\delta) = [1 + i\delta(\rho\Delta_\rho)]^m f(\rho,\theta)$, where m is a positive integer (for the moment) and $(\rho\Delta_\rho)$ is to be interpreted as a single operator. Taking the particular case where $\theta = 0$ and replacing $m\delta$ by θ again, it follows that $f(\rho,\theta) = [1 + i\delta(\rho\Delta_\rho)]^{\theta/\delta} f(\rho,0)$, where θ/δ is an integer.

The continuation operator C_θ is to be defined by

$$C_\theta = [1 + i\delta(\rho\Delta_\rho)]^{\theta/\delta} = \sum_{j=0}^{\theta/\delta} \binom{\theta/\delta}{j} (i\delta)^j (\rho\Delta_\rho)^j \quad \ldots\ldots(5.3.1)$$

where $(\rho\Delta_\rho)^j$ denotes the jth iteration of the operator $(\rho\Delta_\rho)$; $(\rho\Delta_\rho)^0 = 1$. For functions defined at points of P on the polar axis, C_θ provides the discrete analytic continuation to a function defined at every other point of P. By construction $f(\rho,\theta) = C_\theta f(\rho,0)$ is the unique discrete analytic function with prescribed values on the polar axis.

An analogue for multiplication of discrete functions f and g can be defined by

$$(f*g)(\rho,\theta) = C_\theta [f(\rho,0)g(\rho,0)]$$

$$= \sum_{j=0}^{\theta/\delta} \binom{\theta/\delta}{j} (i\delta)^j (\rho\Delta_\rho)^j [f(\rho,0)g(\rho,0)] \ldots\ldots(5.3.2)$$

The resulting function is also discrete analytic and by methods similar to those used by Harman [56] in his earlier paper, he proved that $(P,+,*)$ is an integral domain.

4. THE DISCRETE POWER FUNCTION. To develop power series representations for discrete analytic functions, a suitable

analogue of the classical function z^n is needed. The geometric difference operator Δ_ρ has the important property : $\Delta_\rho \rho^n = [n]\,\rho^{n-1}$; where $[n] \equiv (1-q^n)/(1-q)$ is the q-difference analogue of $n([n] \longrightarrow n$ as $q \longrightarrow 1)$. If $z = \rho \exp\{i\theta\}$, where $z \in P$, the discrete power $z^{(n)}$ will be defined by means of the discrete continuation of ρ^n into the lattice P. Hence

$$z^{(n)} = (\rho,\theta)^{(n)} \equiv C_\theta(\rho^n) = \sum_{j=0}^{\theta/\delta} \binom{\theta/\delta}{j} (i\delta)^j (\rho\Delta_\rho)^j (\rho^n)$$

Now since $(\rho\Delta_\rho)\,\rho^n = [n]\,\rho^n$; it follows that $(\rho\Delta_\rho)^j \rho^n = [n]^j \rho^n$, and

$$z^{(n)} = \rho^n \sum_{j=0}^{\theta/\delta} \binom{\theta/\delta}{j} (i\delta)^j [n]^j = \rho^n (1 + i\delta[n])^{\theta/\delta}$$

Define $e_q^\delta(n,\theta) = (1 + i\delta[n])^{\theta/\delta}$; the above then becomes

$$z^{(n)} = (\rho,\theta)^{(n)} = \rho^n e_q^\delta(n,\theta) \qquad \text{......(5.4.1)}$$

By construction, z^n is discrete analytic and satisfies further conditions which illustrate suitability as an analogue of z^n. For example, the following additional properties can readily be verified:

(i) $\Delta_\rho z^{(n)} = [n]\,\rho^{n-1} e_q^\delta(n,\theta)$; $\Delta_\theta z^{(n)} = i[n]\,\rho^n e_q^\delta(n,\theta)$;

(ii) $z^{(o)} = 1$;

(iii) for $n > 0$; $z^{(n)} = 0$ if and only if $z = 0$;

(iv) $z^{(n)} * z^{(m)} = z^{(n+m)}$, where $*$ is defined by (5.3.2).

Property (iv) was evident for the q-analytic function analogues of z^n considered by Harman [57], but property (iii) is new for pseudo-powers in discrete analytic function theories. One shortcoming of the present analogue, however, is not in general $z^{(1)} \neq z$.

A factorization result was found by Harman [55] for q-analytic polynomials. An analogue of the fundamental theorem of algebra can also be demonstrated for polynomials discrete analytic in the polar sense. A discrete polynomial is to be defined as $p_n(z) = \sum_{j=0}^{n} a_j z^{(j)}$, where a_j are constants with $a_o = 1$. The corresponding classical polynomial is to be denoted by $f_n(z) = \sum_{j=0}^{n} a_j z^j$.

THEOREM 5.4.1. If p_n is a discrete polynomial then

$$p_n(z) = (z^{(1)} - \alpha_1)*(z^{(1)} - \alpha_2)* \ldots\ldots\ldots *(z^{(1)} - \alpha_n),$$

where α_i, $i = 1,2,\ldots\ldots n$ are the zeros of the corresponding polynomial f_n.

5. DISCRETE POWER SERIES REPRESENTATION. In the above development of the discrete function $z^{(n)}$, if n is permitted to take any value then $z^{(n)}$ is still well defined by (5.4.1) and satisfies the properties outlined in Section 4 above. This represents an improvement over the monodiffric analogue of z^n studied by Harman [53], where n could not take negative integer values, and so discrete analogues of Laurent series now become possible.

Since there were no bound restrictions placed on the values of θ in the definition of a discrete analytic function given by (5.2.3) and in the continuation operator C_θ, the function $z^{(n)}$ would in general be multi-valued. For convenience, a function f, defined on the lattice P, will be said to be discrete entire on P if (5.2.3), is satisfied, where $\theta = j\delta$, $j = 0,1, \ldots\ldots. M-2$. By the development in Section 3 above, it can then readily be verified that discrete entire functions f can be specified at each point of $P(\theta = j\delta$, $j=0,1, ,\ldots,. M-1)$ by the unique representation $f(z) = C_\theta f(\rho ,0)$. The discrete entire function $\underset{\sim}{z}^{(n)}$ is then single valued and denoted by $z^{(n)} = \rho^n e_q^\delta(n,\theta)$, where $\theta = 0,\delta,2\delta \ldots\ldots\ldots 2\pi -\delta$. For the remainder of this chapter, Θ will be restricted to this cycle so that single valued functions will be the theme.

In the theory of discrete analytic analytic functions on a square lattice, Zeilberger [129] showed that it not possible to replace z^n in an entire function $\sum_{n=0}^{\infty} a_n z^n$ by a discrete pseudo-power counterpart and still guarantee convergence. However, this is possible in the present theory as the following theorem and corollary demonstrate.

THEOREM 5.5.1. If $\sum_{n=0}^{\infty} a_n \rho^n$ converges for all values of ρ such that $(\rho,0) \in P$, then $C_\Theta \sum_{n=0}^{\infty} a_n \rho^n = \sum_{n=0}^{\infty} a_n z^{(n)}$ with convergence for all $z = (\rho,\Theta)$ in P.

COROLLARY 5.5.1. If $\sum_{n=0}^{\infty} a_n z^n$ converges for all z in C then $\sum_{n=0}^{\infty} a_n z^{(n)}$ converges for all z in P.

Theorem 5.5.1 has a corresponding result for negative powers and the following can be readily demonstrated using a similar proof.

THEOREM 5.5.2. If $\sum_{n=0}^{\infty} a_n \rho^{-n}$ converges for all values of ρ such that $(\rho,\Theta) \in P$, then $C_\Theta \sum_{n=0}^{\infty} a_n \rho^{-n} = \sum_{n=0}^{\infty} a_n z^{(-n)}$ with convergence for all $z = (\rho,\Theta)$ in P.

The following theorem indicates the most important property of the discrete powers $z^{(n)}$ they form a basis for discrete functions on the polar lattice P.

THEOREM 5.5.3. If f is any discrete entire function on P, then it has a convergent power series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^{(n)} \quad \text{for all values of } z \text{ in } P.$$

Under certain circumstances the discrete Laurent series in Theorem 5.5.3 may reduce to a series of positive discrete powers. For example, if values of the discrete entire function, at lattice points on the polar axis, are known to be representable as a Taylor series about the origin then

$$f(z) = C_\theta\, f(\rho) = C_\theta \sum_{n=0}^{\infty} f^{(n)}(0)\, \rho^n/n!$$

and so by Theorem 5.5.1

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(0)\, z^{(n)}/n! \ .$$

Also it is not difficult to verify that

$$f^{(n)}(0) = n!\, \Delta_\rho^n\, f(0)/[n] \ ,$$

where $\Delta_\rho^n f(0) = \lim_{\delta \to 0} \Delta_\rho^n f(\rho)$ and $[n]! = [n][n-1] \ldots\ldots [1]$, and hence, it follows that

$$f(z) = \sum_{n=0}^{\infty} \Delta_{\rho}^{n} f(0) z^{(n)} / [n]! ,$$

the discrete analytic analogue of Taylor's series about the origin.

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